

# On the cloaking effects associated with anomalous localized resonance

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Regions of anomalous localized resonance, such as occurring near superlenses, are shown to lead to cloaking effects. This occurs when the resonant field generated by a polarizable line or point dipole acts back on the polarizable line or point dipole and effectively cancels the field acting on it from outside sources. Cloaking is proved in the quasistatic limit for finite collections of polarizable line dipoles that all lie within a specific distance from a coated cylinder having a shell permittivity  $\epsilon_s \approx -\epsilon_m \approx -\epsilon_c$  where  $\epsilon_m$  is the permittivity of the surrounding matrix, and  $\epsilon_c$  is the core permittivity. Cloaking is also shown to extend to the Veselago superlens outside the quasistatic regime: a polarizable line dipole located less than a distance  $d/2$  from the lens, where  $d$  is the thickness of the lens, will be cloaked due to the presence of a resonant field in front of the lens. Also a polarizable point dipole near a slab lens will be cloaked in the quasistatic limit.

**Keywords:** cloaking; localized resonance; superlenses; negative refraction

## 1. Introduction

Nicorovici *et al.* (1994) found that a coated cylinder, now called a cylindrical superlens, with a core of dielectric constant  $\epsilon_c = 1$  and radius  $r_c$  and a shell with dielectric constant  $\epsilon_s = -1 + i\epsilon_s''$  and outer radius  $r_s$  would in the limit  $\epsilon_s'' \rightarrow 0$  be invisible to any applied quasistatic transverse magnetic (TM) field. Here we show that not only is the lens invisible in this limit, but so too are cylindrical objects, or at least any finite collection of polarizable line dipoles, that lie within a radius  $r_{\#} \equiv \sqrt{r_s^3/r_c}$  of the cylindrical superlens.

In that paper some other remarkable properties were found to hold in the limit  $\epsilon_s'' \rightarrow 0$ . First a cylinder of radius  $r_c$  and permittivity  $\epsilon_c \neq 1$  placed inside the cylindrical shell would to an outside observer appear magnified by a factor of  $h = r_s^2/r_c^2$  and respond like a solid cylinder of permittivity  $\epsilon_c$  of radius  $r_* = r_c h = r_s^2/r_c$  to any quasistatic TM applied fields that do not have sources within the radius  $r_*$ . Second, again when  $\epsilon_c \neq 1$ , a dipole line source positioned outside the coated cylinder at a radius  $r_0$  less than  $r_{\text{crit}} = r_s^3/r_c^2 = r_*^2/r_s$  would have an image

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dipole line source (ghost source) lying *outside* the cylinder at the radius  $\tilde{r}_0 = r_*^2/r_0 > r_s$ . Specifically, because there cannot be singularities in the field outside the cylinder, apart from the original line dipole source, at radii greater than  $\tilde{r}_0$  the image dipole was found to appear to be like an actual line source with the approximation becoming better as  $\varepsilon_s'' \rightarrow 0$  but inside the radius  $\tilde{r}_0$  numerical computations showed that the field had enormous oscillations, which grew as  $\varepsilon_s'' \rightarrow 0$ .

A mathematically very similar phenomena was implied by the bold claim of Pendry (2000) (see also the reviews of Pendry (2004) and Ramakrishna (2005)) that the Veselago slab lens (Veselago 1968), consisting of a slab of material having thickness  $d$ , relative electric permittivity  $\varepsilon_s = -1$ , relative magnetic permeability  $\mu_s = -1$ , would act as a superlens: a line (or point) dipole source located at distance  $d_0$  in front of the Veselago lens would when  $d_0 < d$ , have a line image dipole source (ghost source) lying *outside* the lens at a distance  $d - d_0$  from the back of the lens. Again there cannot be singularities in the field lying outside the lens apart from the original line, or point, dipole source as emphasized by Maystre & Enoch (2004) among others. For the lossless Veselago lens Garcia & Nieto-Vesperinas (2002) and Pokrovsky & Efros (2002) claimed the fields lost their square integrability throughout a layer of thickness  $2(d - d_0)$  centered on the back interface, although it is not clear to us whether the claimed divergence within the entire layer in the lossless case is an artifact of the use of plane wave expansions, in the same way that Taylor series diverge outside the radius of convergence, but other expansions have different regions of convergence. One should allow for some small loss, taking  $\varepsilon_s = -1 + i\varepsilon_s''$ ,  $\mu_s = -1 + i\mu_s''$  and consider what happens when  $\varepsilon_s''$  and  $\mu_s''$  are very small. At distances greater than  $d - d_0$  from the back of the lens the image source appears to be like an actual line (or point) source with the approximation becoming better as  $(\varepsilon_s'', \mu_s'') \rightarrow 0$  but at distances less than  $d - d_0$  from the back of the lens the field has enormous oscillations, which grow as  $(\varepsilon_s'', \mu_s'') \rightarrow 0$ . Contrary to the conventional picture, the field also has enormous oscillations in front of the lens as shown by Podolskiy *et al.* (2005) and these fields are the ones responsible for cloaking (see also Rao & Ong (2003), Shvets (2003), Merlin (2004) and Guenneau *et al.* (2005)) whose investigations provided some evidence of large fields in front of the lens). We will see here that in fact the field generated by a constant amplitude line dipole source diverges as  $\varepsilon_s'', \mu_s'' \rightarrow 0$  within a distance of  $d - d_0$  from either the front or back interface. This generalizes the result of Milton *et al.* (2005) where the same regions of field divergence were found for the quasistatic equations. As that paper will be frequently referenced, it will be denoted by the acronym MNMP. We will find that a polarizable line dipole less than a distance  $d/2$  from the Veselago lens becomes cloaked in the limit as  $(\varepsilon_s'', \mu_s'') \rightarrow 0$ . Furthermore we will see that, in the quasistatic limit, a polarizable point dipole outside a slab having electric permittivity  $\varepsilon_s = -1 + i\varepsilon_s''$ , and any magnetic permeability, becomes cloaked as  $\varepsilon_s'' \rightarrow 0$ .

Following Milton (2002), §11.7, and MNMP we say an inhomogeneous body exhibits anomalous localized resonance if as the loss goes to zero (or for static problems, as the system of equations lose ellipticity) the field magnitude diverges to infinity throughout a specific region with *sharp* boundaries *not* defined by any discontinuities in the moduli, but the field converges to a smooth

field outside that region. A region where the field diverges will be called a region of local resonance. We will see that cloaking occurs when a polarizable line or point dipole interacts with the resonant field that is generated by the polarizable line or point dipole itself, and that the effect of the resonant region is to cancel the field acting on the polarizable line or point dipole from outside sources. A region where a polarizable line or point dipole is cloaked will be called a cloaking region.

Cloaking can also be regarded as a consequence of energy considerations: if the polarizable line or point dipole was not cloaked then the energy sources in the system would have to be infinite. As shown in the paper [MNMP](#), and as stemmed from a suggestion of Alexei Efros (2005, personal communication), a line dipole source with a fixed dipole moment less than a distance  $d/2$  from the slab lens would could cause infinite energy loss. (We will see this is true not just for the quasistatic solution, but also for the solution to Maxwell's equations for the Veselago lens.) Any realistic dipole source (such as a polarizable line source) within this region must have a dipole moment which vanishes as  $\epsilon_s'' \rightarrow 0$ . As the lens provides a perfect image of the source in the limit  $\epsilon_s'' \rightarrow 0$ , at least further than a distance  $d$  from the slab, we conclude that the dipole source will have a vanishingly small effect on the field further than a distance  $d$  from the slab.

We remark that the cloaking effects discussed here extend to static magnetoelectric equations, as implied by the equivalence discussed in §6 of [MNMP](#) that follows from earlier work of [Cherkaev & Gibiansky \(1994\)](#) and [Milton \(2002\)](#), §11.6. There it is shown that the two-dimensional quasistatic dielectric equations in any geometry (and with possible source terms) can be transformed to a set of magnetoelectric equations (with corresponding source terms) with a symmetric real positive definite tensor entering the constitutive law. Therefore properties like superlensing and cloaking which hold for the quasistatic dielectric equations automatically also hold for the equivalent magnetoelectric equations.

## 2. Some simple examples of cloaking in the quasistatic limit

First we present an example which shows that a polarizable line with polarizability  $\alpha$  (and possibly a source term) can be cloaked when immersed in a TM field surrounding a coated cylinder with inner radius  $r_c$ , outer radius  $r_s$ , and with cylinder axis  $x = y = 0$ . The polarizable line is placed along  $x = r_0$  and  $y = 0$ , where  $r_0 > r_s$ . Suppose  $(E_x(x, y), E_y(x, y), 0)$  is the field without the polarizable line present due to:

- (i) Fields generated by fixed sources not varying in the  $Z$  direction lying outside the radius  $r_* \equiv r_s^2/r_c$  when  $\epsilon_c \neq \epsilon_m$ , and the radius  $r_{\#} \equiv \sqrt{r_s^3/r_c}$  when  $\epsilon_c = \epsilon_m$ . (Here we use  $Z$  for the  $z$ -coordinate to avoid confusion with  $z = x + iy$ ). We assume these sources are not perturbed when the polarizable line is introduced.
- (ii) Fields generated by the coated cylinder due to its interaction with these fixed sources.

Let  $\varepsilon_m$ ,  $\varepsilon_s$ , and  $\varepsilon_c$  denote the permittivity in the matrix, shell and core, and let us set

$$\eta = \frac{\varepsilon_m - \varepsilon_c}{\varepsilon_m + \varepsilon_c}, \quad \frac{(\varepsilon_s + \varepsilon_c)(\varepsilon_m + \varepsilon_s)}{(\varepsilon_s - \varepsilon_c)(\varepsilon_m - \varepsilon_s)} = \delta e^{i\phi}, \quad (2.1)$$

where  $\delta$  is real and positive. We assume that  $\varepsilon_c$  and  $\varepsilon_m$  remain fixed with  $\varepsilon_m$  real and positive, and with  $\varepsilon_c$  possibly complex (with non-negative imaginary part) but not real and negative, and that  $\varepsilon_s$  approaches  $-\varepsilon_m$  along a trajectory in the upper half of the complex plane in such a way that  $\delta \rightarrow 0$  but  $\phi$  remains fixed.

When  $\varepsilon_s$  is close to  $-\varepsilon_m$  equation (2.1) implies

$$\left. \begin{aligned} \varepsilon_s &\approx (-1 + 2\delta e^{i\phi}/\eta)\varepsilon_m && \text{when } \varepsilon_c \neq \varepsilon_m, \\ &\approx (-1 + 2i\sqrt{\delta}e^{i\phi/2})\varepsilon_m && \text{when } \varepsilon_c = \varepsilon_m, \end{aligned} \right\} \quad (2.2)$$

and so we have

$$\left. \begin{aligned} \delta &\approx |\varepsilon_s + \varepsilon_m| |\eta| / (2\varepsilon_m) && \text{when } \varepsilon_c \neq \varepsilon_m, \\ &\approx |\varepsilon_s + \varepsilon_m|^2 / (4\varepsilon_m^2) && \text{when } \varepsilon_c = \varepsilon_m. \end{aligned} \right\} \quad (2.3)$$

Thus, for small  $\delta$  the trajectory approaches  $-\varepsilon_m$  in such a way that the argument of  $\varepsilon_s + \varepsilon_m$  is approximately constant.

Let us drop the  $Z$  field component of the electric field since it is zero for TM fields. The field  $(E_x^0, E_y^0)$  acting on the polarizable line has two components:

$$(E_x^0, E_y^0) = (E_x + E_x^r, E_y + E_y^r), \quad (2.4)$$

where

$$\left. \begin{aligned} E_x &\equiv E_x(r_0, 0), \quad E_y \equiv E_y(r_0, 0), \quad E_x^r \equiv E_x^r(r_0, 0), \quad E_y^r \equiv E_y^r(r_0, 0), \\ (E_x^r(x, y), E_y^r(x, y)) &= (-\partial V_{\text{in}}(x, y)/\partial x, -\partial V_{\text{in}}(x, y)/\partial y), \end{aligned} \right\} \quad (2.5)$$

and  $V_{\text{in}}(x, y)$  is the (possibly resonant) response potential in the matrix generated by the coated cylinder responding to the polarizable line itself (not including the field generated by the coated cylinder responding to the other fixed sources). The field  $(E_x^r, E_y^r)$  must depend linearly on the dipole moment of the polarizable line, and in fact, as we will see shortly, this dependence has the form

$$\begin{pmatrix} E_x^r \\ E_y^r \end{pmatrix} = c(\delta) \begin{pmatrix} k^e \\ -k^o \end{pmatrix}, \quad (2.6)$$

in which, following the notation of MNMP,  $k^e$  and  $k^o$  are the (suitably normalized) dipole moments of the polarizable line ( $k^e$  gives the amplitude of the dipole component which has even symmetry about the  $x$ -axis while  $k^o$  gives the amplitude of the dipole component which has odd symmetry about the  $x$ -axis). We will see that  $|c(\delta)|$  can diverge to infinity as  $\delta \rightarrow 0$ , and that when this happens the polarizable line becomes cloaked. Figure 1 shows the cloaking with a

fixed dipole line source acting on a polarizable line which is in the cloaking region of the cylindrical superlens. For comparison [figure 2](#) has the polarizable line outside the cloaking region.

Now if  $\alpha$  denotes the polarizability tensor of the line (which need not be assumed proportional to  $\mathbf{I}$ ) and we allow for the fact that the polarizable line could have a fixed dipole source term, then we have

$$\begin{pmatrix} k^e \\ -k^o \end{pmatrix} = \alpha \begin{pmatrix} E_x^0 \\ E_y^0 \end{pmatrix} + \begin{pmatrix} k_0^e \\ -k_0^o \end{pmatrix}, \quad (2.7)$$

where the source terms  $k_0^e$  and  $k_0^o$  are assumed to be fixed. This implies

$$\begin{pmatrix} k^e \\ -k^o \end{pmatrix} = \alpha \begin{pmatrix} E_x \\ E_y \end{pmatrix} + \alpha c(\delta) \begin{pmatrix} k^e \\ -k^o \end{pmatrix} + \begin{pmatrix} k_0^e \\ -k_0^o \end{pmatrix}, \quad (2.8)$$

which when solved for the dipole moment  $(k^e, -k^o)$  gives

$$\begin{pmatrix} k^e \\ -k^o \end{pmatrix} = \alpha_* \begin{pmatrix} E_x \\ E_y \end{pmatrix} + \begin{pmatrix} k_*^e \\ -k_*^o \end{pmatrix}, \quad (2.9)$$

where

$$\alpha_* = [\alpha^{-1} - c(\delta)\mathbf{I}]^{-1}, \quad \begin{pmatrix} k_*^e \\ -k_*^o \end{pmatrix} = [\mathbf{I} - c(\delta)\alpha]^{-1} \begin{pmatrix} k_0^e \\ -k_0^o \end{pmatrix}, \quad (2.10)$$

are the ‘effective polarizability tensor’ and ‘effective source terms’.

Notice that when  $|c(\delta)|$  is very large we have

$$\alpha_* \approx \frac{-\mathbf{I}}{c(\delta)}, \quad \begin{pmatrix} k_*^e \\ -k_*^o \end{pmatrix} \approx \frac{-\alpha^{-1}}{c(\delta)} \begin{pmatrix} k_0^e \\ -k_0^o \end{pmatrix}. \quad (2.11)$$

So in this limit the effective polarizability tensor has a very weak dependence on  $\alpha$  (unless  $\alpha$  has one or more very small eigenvalues) while the effective source term has a strong dependence on  $\alpha$ ! Both expressions tend to zero as  $|c(\delta)| \rightarrow \infty$ , which explains why cloaking occurs.

It is instructive to see what happens to the local field  $(E_x^0, E_y^0)$  acting on the polarizable line as  $|c(\delta)| \rightarrow \infty$ . For simplicity let us suppose  $k_0^e = k_0^o = 0$  and  $\alpha = \alpha\mathbf{I}$ . Then from equations (2.4), (2.6), (2.9) and (2.10) we see that

$$E_x^0 = E_x + c(\delta)k^e = E_x + \frac{c(\delta)E_x}{\alpha^{-1} - c(\delta)} = \frac{E_x}{1 - \alpha c(\delta)}, \quad (2.12)$$

goes to zero as  $|c(\delta)| \rightarrow \infty$ , and similarly so too does  $E_y^0$ . This explains why the ‘effective polarizability’ vanishes as  $|c(\delta)| \rightarrow \infty$ : the effect of the resonant field is to cancel the field  $(E_x, E_y)$  acting on the polarizable line.

To obtain an explicit expression for  $c(\delta)$  we have from equations (2.5), (3.9) and (3.10) of [MNMP](#) that

$$V_{\text{in}}(x, y) = [f_{\text{in}}^e(z) + f_{\text{in}}^e(\bar{z})]/2 + [f_{\text{in}}^o(z) - f_{\text{in}}^o(\bar{z})]/(2i), \quad (2.13)$$

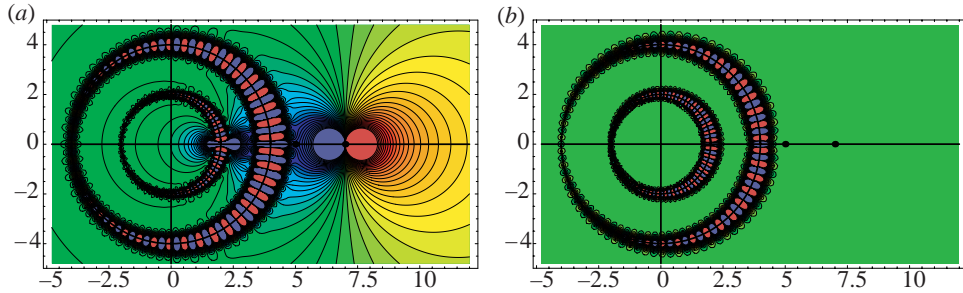


Figure 1. Numerical computations of equipotentials for (a)  $\text{Re}[V(z)]$  and (b)  $\text{Im}[V(z)]$  for a cylindrical lens with  $\epsilon_m = \epsilon_c = 1$  and  $\epsilon_s = -1 + 10^{-12}i$ , and with  $r_c = 2$ ,  $r_s = 4$ ,  $r_* = 8$  and  $r_\# = 5.66$ . There is a fixed dipole source with  $(k_1^e, k_1^o) = (1, 0)$  at  $z = 7$ , chosen to be between  $r_*$  and  $r_\#$ . There is a polarizable dipole with  $\alpha = 2$  and  $(k_0^e, k_0^o) = 0$  at  $z = r_0 = 5$  chosen to be in the cloaking region (located at the point just to the right of the outermost resonant region). In both figures both the coated cylinder and the polarizable dipole are essentially invisible outside the cloaking region and do not disturb the dipole potential surrounding the fixed source. The computations show that the polarizable dipole has a very small moment  $k^e = -6.31 \times 10^{-7} + 2.30 \times 10^{-6}i$  and  $k^o = 0$ . The solid red regions are below a low cutoff equipotential, while the solid blue regions are above a high cutoff equipotential.

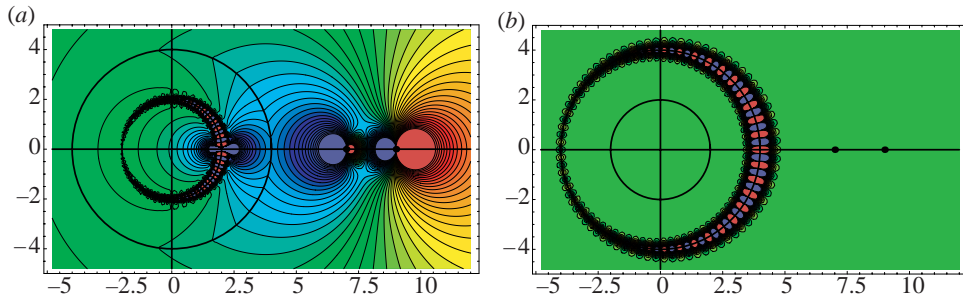


Figure 2. Same as for figure 1 except the fixed dipole source and the polarizable dipole have been moved to the right by 2 units to  $r_1 = 9$  and to  $r_0 = 7$ , respectively. Equivalently, the cylindrical lens has been moved to the left by 2 units while keeping the fixed dipole source and the polarizable dipole in the same position. The polarizable dipole is now visible because it is outside the cloaking region. The computations show that the polarizable dipole has a large moment  $k^e = 0.5 + 8.39 \times 10^{-8}i$  and  $k^o = 0$ .

where  $z = x + iy$  and for  $p = e, o$

$$f_{\text{in}}^p(z) = -\frac{qk^p S(\delta, r_*^2/(r_0 z))}{r_0 \eta_{\text{sc}}} - \frac{qk^p \delta e^{i\phi} \eta_{\text{sc}} S(\delta, r_s^2/(r_0 z))}{r_0}, \quad (2.14)$$

in which  $q = 1$  for  $p = e$  and  $q = -1$  for  $p = o$  and

$$\eta_{\text{sc}} \equiv \frac{\epsilon_s - \epsilon_c}{\epsilon_s + \epsilon_c}, \quad S(\delta, w) \equiv \sum_{\ell=1}^{\infty} \frac{w^\ell}{1 + \delta e^{i\phi} h^\ell}, \quad h = \frac{r_s^2}{r_c^2}. \quad (2.15)$$

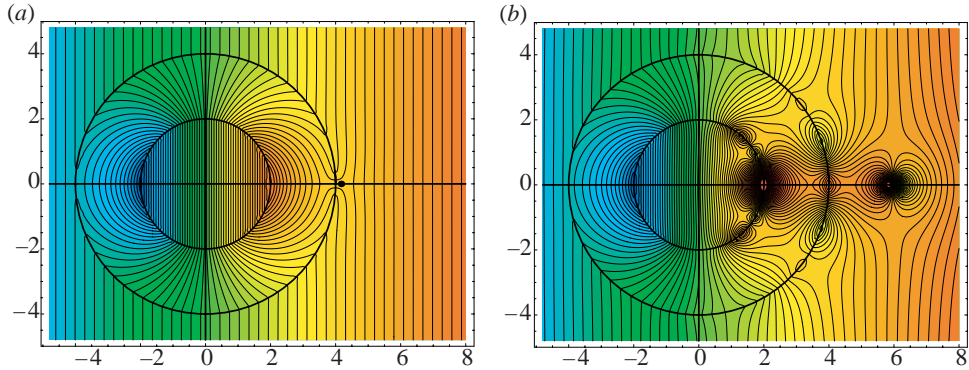


Figure 3. Numerical computations of equipotentials for  $\text{Re}[V(z)]$  for a cylindrical lens with  $\varepsilon_m = \varepsilon_c = 1$  and  $\varepsilon_s = -1 + 0.01i$ , and with  $r_c = 2$ ,  $r_s = 4$ ,  $r_* = 8$ , and  $r_\# = 5.66$ . A uniform field  $\mathbf{E} = (-1, 0)$  acts on the system. In the figure on the left the polarizable line dipole with  $\alpha = 2$  and  $(k_0^e, k_0^o) = 0$  is located close to the lens at  $r_0 = 4.166$  and it along with the cylindrical lens are essentially invisible to the uniform field. The computations show that the polarizable line dipole has  $k^e = 0.000012 - 0.00066i$  and  $k^o = 0$ . In the figure on the right the polarizable line dipole is located outside the cloaking region at  $r_0 = 5.95$  and significantly perturbs the uniform field. The computations show that the polarizable line dipole has  $k^e = -1.68 - 0.74i$  and  $k^o = 0$ .

Differentiating equation (2.13) gives

$$\left. \begin{aligned} E_x^r(x, y) &= -[f_{\text{in}}^{e'}(z) + f_{\text{in}}^{e'}(\bar{z})]/2 - [f_{\text{in}}^{o'}(z) - f_{\text{in}}^{o'}(\bar{z})]/(2i), \\ E_y^r(x, y) &= -i[f_{\text{in}}^{e'}(z) - f_{\text{in}}^{e'}(\bar{z})]/2 - [f_{\text{in}}^{o'}(z) + f_{\text{in}}^{o'}(\bar{z})]/2, \end{aligned} \right\} \quad (2.16)$$

where

$$f_{\text{in}}^{\text{p}'}(z) = \text{d}f_{\text{in}}^{\text{p}}(z)/\text{d}z = \frac{qk^{\text{p}}r_*^2S'(\delta, r_*^2/(r_0z))}{r_0^2z^2\eta_{\text{sc}}} + \frac{qk^{\text{p}}r_s^2\delta e^{i\phi}\eta_{\text{sc}}S'(\delta, r_s^2/(r_0z))}{r_0^2z^2}, \quad (2.17)$$

in which

$$S'(\delta, w) \equiv \frac{\text{d}S(\delta, w)}{\text{d}w} = \sum_{\ell=1}^{\infty} \frac{\ell w^{\ell-1}}{1 + \delta e^{i\phi} h^\ell}. \quad (2.18)$$

These expressions simplify if  $z$  is real since then  $f_{\text{in}}^{\text{p}'}(z) - f_{\text{in}}^{\text{p}'}(\bar{z}) = 0$  and  $(E_x^r, E_y^r) = (-f_{\text{in}}^{e'}(z), -f_{\text{in}}^{o'}(z))$ . In particular with  $z = r_0$ , we obtain equation (2.6) with

$$c(\delta) = -\frac{r_*^2S'(\delta, r_*^2/r_0^2)}{r_0^4\eta_{\text{sc}}} - \frac{r_s^2\delta e^{i\phi}\eta_{\text{sc}}S'(\delta, r_s^2/r_0^2)}{r_0^4}. \quad (2.19)$$

So far no approximation has been made.

To obtain an asymptotic formula for  $c(\delta)$  when  $\delta$  is small we use the approximations

$$S(\delta, w) \approx \frac{w}{1-w}, \quad S'(\delta, w) \approx \frac{1}{(1-w)^2} \quad \text{for } |w| < 1, \quad (2.20)$$

implied by lemma 3.1 of MNMP, and the approximations

$$\left. \begin{aligned} S(\delta, w) &\approx e^{-\log w \log \delta / \log h} T(w), \\ S'(\delta, w) &\approx -[\log \delta / (w \log h)] e^{-\log w \log \delta / \log h} T(w) + e^{-\log w \log \delta / \log h} T'(w) \\ &\approx -[\log \delta / (w \log h)] e^{-\log w \log \delta / \log h} T(w), \end{aligned} \right\} \quad (2.21)$$

which hold for  $h > |w| > 1$  and are implied by equation (3.22) of MNMP, where

$$T(w) = \sum_{j=-\infty}^{\infty} \frac{w^j}{1 + e^{i\phi} h^j}, \quad T'(w) = \sum_{j=-\infty}^{\infty} \frac{jw^{j-1}}{1 + e^{i\phi} h^j}, \quad (2.22)$$

and in making the last approximation in equation (2.21) we have assumed that  $\delta$  is so small that  $|\log \delta|$  is very large. (It should be stressed that although we are assuming extremely small loss here, the polarizable line dipole can still be cloaked at moderate loss: see figure 3.) Let us first treat the case where  $\varepsilon_c$  is fixed and not equal to  $\varepsilon_m$  and  $r_0 < r_*$ . Then we have  $\eta_{sc} \approx 1/\eta$  and substituting these approximations in equations (2.14) and (2.19) and keeping only the terms which are dominant because  $\delta$  is very small gives, for  $r_*^2/r_0 > |z| > r_s$ ,

$$f_{in}^p(z) \approx -qk^p \eta e^{[\log z - \log(r_*^2/r_0)] \log \delta / \log h} r_0^{-1} T(r_*^2/(r_0 z)), \quad (2.23)$$

which is equivalent to equation (3.33) of MNMP, and implies

$$f_{in}^{p'}(z) \approx \frac{-qk^p \eta \log \delta}{zr_0 \log h} e^{-\log(r_*^2/(zr_0)) \log \delta / \log h} T(r_*^2/(r_0 z)), \quad (2.24)$$

and

$$c(\delta) \approx \frac{\eta \log \delta}{r_0^2 \log h} e^{-2 \log(r_*/r_0) \log \delta / \log h} T(r_*^2/r_0^2). \quad (2.25)$$

We see that  $|c(\delta)| \rightarrow \infty$  as  $\delta \rightarrow 0$  when  $r_0 < r_*$ . Consequently both the ‘effective polarizability tensor’ and the ‘effective source terms’ approach zero in the limit  $\delta \rightarrow 0$ . For simplicity let us suppose  $\alpha = \alpha \mathbf{I}$ . Then when  $\delta$  is very small from equations (2.9) and (2.11) we have

$$k^e \approx [-E_x - \alpha^{-1} k_0^e]/c(\delta), \quad k^o \approx [E_y - \alpha^{-1} k_0^o]/c(\delta). \quad (2.26)$$

Thus, the resonant potential associated with the polarizable line has, from equation (2.23),

$$\begin{aligned} f_{in}^e &\approx [E_x + \alpha^{-1} k_0^e] \eta e^{[\log z - \log(r_*^2/r_0)] \log \delta / \log h} r_0^{-1} T(r_*^2/(r_0 z))/c(\delta) \\ &\approx [E_x + \alpha^{-1} k_0^e] \delta^{\log(r/r_0)/\log h} e^{-2\pi i \theta/\gamma_0} r_0 T(r_*^2/(r_0 z)) \log h / (T(r_*^2/r_0^2) \log \delta), \end{aligned} \quad (2.27)$$

where  $\gamma_0 = -2\pi \log h / \log \delta$  is the angular distance between peaks in the resonant potential and  $z = re^{i\theta}$ . Similarly we have

$$f_{in}^o \approx [E_y - \alpha^{-1} k_0^o] \delta^{\log(r/r_0)/\log h} e^{-2\pi i \theta/\gamma_0} r_0 T(r_*^2/(r_0 z)) \log h / (T(r_*^2/r_0^2) \log \delta). \quad (2.28)$$

Thus as  $\delta \rightarrow 0$  these resonant potentials in the matrix converge to zero in the region  $r > r_0$  but diverge to infinity with increasingly rapid angular oscillations

for  $r_s \leq r < r_0$ . (This is to be contrasted with the resonant potential in the matrix associated with a line dipole having fixed  $k^e$  and  $k^o$ , which as can be seen from equation (2.23) diverges to infinity in the much larger region  $r_s \leq r < r_*^2/r_0$ .) A simple calculation, based on substituting the formulae (2.26) and (2.25) into the formula (3.35) and (3.37) in MNMP for the resonant potentials in the shell and core, shows that the field associated with the polarizable line is resonant in the entire annulus  $r_s^2/r_0 \leq r < r_0$ , and converges to zero outside this annulus.

Suppose the source outside is a line dipole with a fixed source term  $(k_1^e, k_1^o) = (k_1^e, 0)$  located at the point  $(r_1, 0)$ , where  $r_1 > r_* > r_0$ . When  $r_1$  is chosen with  $r_*^2/r_0 > r_1 > r_*$  the polarizable line will be located within the resonant region generated by the line source outside. One might at first think that a polarizable line placed within the resonant region would have a huge response because of the enormous fields there. However, we will see that the opposite is true: the dipole moment of the polarizable line still goes to zero as  $\delta \rightarrow 0$ . From equations (2.6), (2.16) and (2.24), with  $r_0$  replaced by  $r_1$ , the field at the point  $(r_0, 0)$  when the polarizable line is absent will be

$$E_x = c_1(\delta)k_1^e, \quad E_y(x, y) = 0, \quad (2.29)$$

where

$$c_1(\delta) \approx \frac{\eta \log \delta}{r_0 r_1 \log h} e^{-\log(r_*^2/(r_0 r_1)) \log \delta / \log h} T(r_*^2/(r_0 r_1)). \quad (2.30)$$

This and equation (2.26) implies the polarizable line has a dipole moment

$$\begin{aligned} k^e &\approx -[E_x + \alpha^{-1} k_0^e]/c(\delta) \approx -c_1(\delta)k_1^e/c(\delta) \\ &\approx -\frac{r_0 T(r_*^2/(r_0 r_1))}{r_1 T(r_*^2/r_0^2)} \delta^{\log(r_1/r_0)/\log h} k_1^e. \end{aligned} \quad (2.31)$$

So  $k^e$  scales as  $\delta^{\log(r_1/r_0)/\log h}$ , which goes to zero as  $\delta \rightarrow 0$  but fairly slowly when  $r_1$  and  $r_0$  are both close to  $r_*$ . If the source or sources, are outside the critical radius  $r_{\text{crit}} = r_s^3/r_c^2$  then there are no resonant regions associated with these sources and both  $k^e$  and  $k^o$  will scale like  $1/c(\delta)$ , i.e. as  $-\delta^{2 \log(r_*/r_0)/\log h} / \log \delta$ , which goes to zero at a faster rate as  $\delta \rightarrow 0$ , but still slowly when  $r_0$  is close to  $r_*$ . On the other hand when  $r_0$  is close to  $r_s$  we have  $r_*/r_0 \approx \sqrt{h}$  and this latter scaling is approximately  $-\delta/\log \delta \sim -\epsilon_s''/\log \epsilon_s''$  which is quite fast.

Let us examine more closely what happens when  $r_0$  approaches  $r_s$  while keeping  $\delta$  fixed (here  $\delta$  is not necessarily small). Then  $r_*^2/r_0^2$  is close to  $h$  and the series  $S'(\delta, r_*^2/r_0^2)$  is close to diverging for any fixed  $\delta$ . When  $\zeta = w/h$  is close to 1 then equation (2.18) implies

$$S'(\delta, w) \approx \frac{1}{\delta e^{i\phi} h} \sum_{\ell=1}^{\infty} \ell \zeta^{\ell-1} = \frac{1}{\delta e^{i\phi} h} \frac{d}{d\zeta} \sum_{\ell=1}^{\infty} \zeta^{\ell} = \frac{\eta_{\text{sc}}(\epsilon_m - \epsilon_s)}{(\epsilon_m + \epsilon_s)(1 - \zeta)^2 h}, \quad (2.32)$$

and as a result from equation (2.19) we have

$$c(\delta) \approx -\frac{r_*^2 S'(\delta, r_*^2/r_0^2)}{r_0^4 \eta_{\text{sc}}} \approx \frac{-r_s^2(\epsilon_m - \epsilon_s)}{(\epsilon_m + \epsilon_s)(r_0^2 - r_s^2)^2} \approx \frac{-(\epsilon_m - \epsilon_s)}{4(\epsilon_m + \epsilon_s)(r_0 - r_s)^2}, \quad (2.33)$$

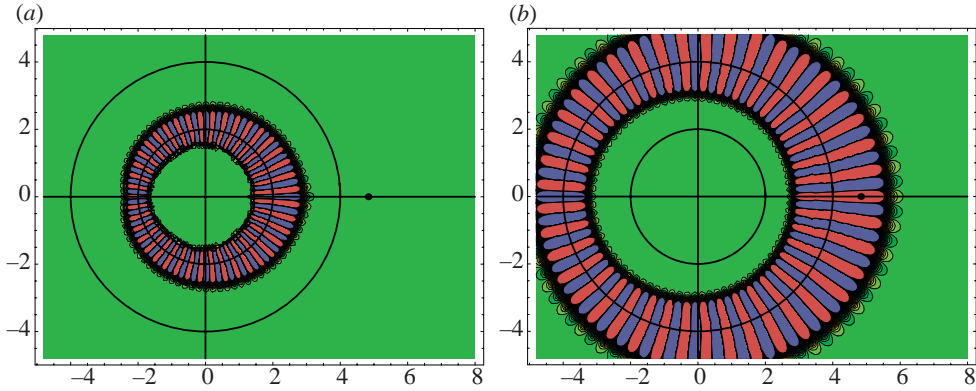


Figure 4. Numerical computations of equipotentials for (a)  $\text{Re}[V(z)]$  and (b)  $\text{Im}[V(z)]$  for a cylindrical lens with  $\varepsilon_m = \varepsilon_c = 1$  and  $\varepsilon_s = -1 + 10^{-12}i$ , and with  $r_c = 2$ ,  $r_s = 4$ ,  $r_* = 8$ , and  $r_\# = 5.66$ . There is a constant energy source with  $k^e = 1/\sqrt{\text{Im}(c(\delta))} = 0.00066$  and  $k^o = 0$  at  $z = r_0 = (r_s + r_\#)/2 = 4.83$ , chosen to be midway between  $r_s$  and  $r_\#$ . The fields are very small outside the cloaking region.

which diverges as  $r_0 \rightarrow r_s$  and is asymptotically independent of  $r_c$  and  $\varepsilon_c$ . Thus, the polarizable line dipole becomes cloaked so long as  $\varepsilon_s \neq \varepsilon_m$ . This cloaking is not due to anomalous localized resonance, as can be seen by considering a polarizable line or point dipole in a material of permittivity  $\varepsilon_m$  outside a half-space filled with material having relative permittivity  $\varepsilon_s$ . In this system the cloaking is due to the interaction of the polarizable line dipole with its image line dipole, and the effect is magnified when  $\varepsilon_s$  is close to  $-\varepsilon_m$ .

The asymptotic analysis is basically similar when  $\varepsilon_c = \varepsilon_m$  and  $r_0 < r_\# \equiv \sqrt{r_s^3/r_c}$ . Then  $\eta_{sc} \approx ie^{-i\phi/2}/\sqrt{\delta}$  and from equations (2.14), (2.19), and (2.21) we have

$$f_{in}^p(z) \approx i q k^p e^{i\phi/2} e^{[\log z - \log(r_* r_s/r_0)] \log \delta / \log h} r_0^{-1} T(r_*^2/(r_0 z)), \quad (2.34)$$

which is equivalent to equation (3.34) of MNMP, and

$$c(\delta) \approx \frac{-ie^{i\phi/2} \log \delta}{r_0^2 \log h} e^{-\log(r_* r_s/r_0^2) \log \delta / \log h} T(r_*^2/r_0^2), \quad (2.35)$$

which diverges as  $\delta \rightarrow 0$  when  $r_0 < r_\#$ . When all the sources lie outside the critical radius  $r_*$  so they do not generate any resonant regions in the absence of the polarizable line, both  $k^e$  and  $k^o$  will scale as  $1/c(\delta)$ , i.e. as  $\delta^{\log(r_* r_s/r_0^2)/\log h / \log \delta}$ , as  $\delta \rightarrow 0$ . When  $r_0$  is close to  $r_s$  we have  $r_* r_s/r_0^2 \approx \sqrt{h}$  and this latter scaling is approximately  $-\sqrt{\delta}/\log \delta \sim -\varepsilon_s''/\log \varepsilon_s''$ , which is the same as when  $\varepsilon_c \neq \varepsilon_m$ . Figure 3a shows the cloaking with uniform field acting on a polarizable line which is close to the cylindrical superlens. For comparison figure 3b has the same polarizable line outside the cloaking region. By substituting equation (2.26) in equation (2.34) we obtain

$$\begin{aligned} f_{in}^e(z) &\approx -[E_x + \alpha^{-1} k_0^e] i e^{i\phi/2} e^{[\log z - \log(r_* r_s/r_0)] \log \delta / \log h} r_0^{-1} T(r_*^2/(r_0 z)) / c(\delta) \\ &\approx [E_x + \alpha^{-1} k_0^e] \delta^{\log(r/r_0)/\log h} e^{-2\pi i \theta / \gamma_0} r_0 T(r_*^2/(r_0 z)) \log h / (T(r_*^2/r_0^2) \log \delta), \end{aligned} \quad (2.36)$$

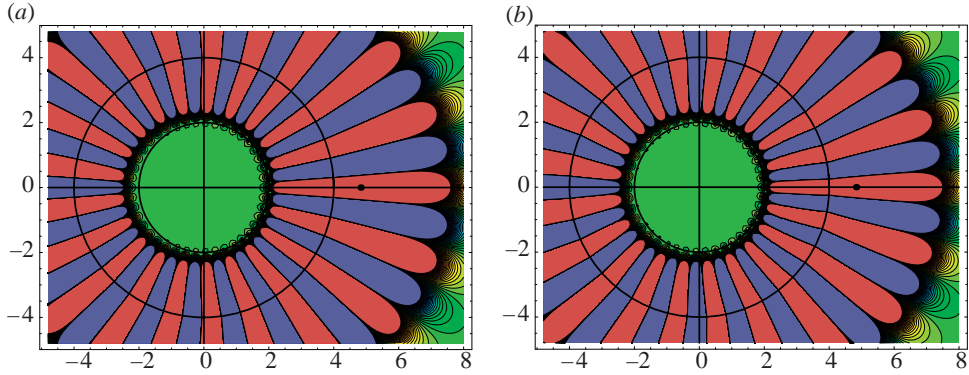


Figure 5. Same as for figure 4 except now the coated cylinder has  $\varepsilon_c = 3$  giving  $k^e = 1/\sqrt{\text{Im}(c(\delta))} = 0.000022$ . Again the fields are very small outside the cloaking region.

which is the same final expression as in equation (2.27). Likewise equation (2.28) still holds. It follows from these expressions and similar analysis based on equation (3.36) and (3.37) of MNMP that as  $\delta \rightarrow 0$  the resonant potentials diverge with increasingly rapid oscillations in the two non-overlapping annuli  $r_0 > r > r_s^2/r_0$  and  $r_c r_0/r_s > r > r_c r_s/r_0$ . Outside these annuli the field converges to the field generated by the fixed sources.

As another interesting example, let us suppose that a single line dipole energy source with, for simplicity,  $k^o = 0$  is placed in the cloaking region, and that  $k^e$  is real and adjusted so that the electrical power  $W_0$  dissipated in the coated cylinder remains constant as the loss goes to zero. Specifically using the identity (4.10) in MNMP we keep

$$W_0 = (\omega/2) \int_{|z| \leq r_s} dx dy \varepsilon'' \mathbf{E}(x, y) \cdot \overline{\mathbf{E}(x, y)} = \omega \pi \varepsilon_m k^e \text{Im}[E_x^0], \quad (2.37)$$

fixed, where the two-dimensional integral is over the area of the coated cylinder (where the loss is) and  $E_x^0$  is the local field acting on the source, which because there are no other sources present is just field  $E_x^r$  generated by the dipole source alone. Since from equation (2.6) we have that  $E_x^r = c(\delta)k^e$ , we deduce that

$$k^e = \sqrt{W_0 / \{\omega \pi \varepsilon_m \text{Im}[c(\delta)]\}}. \quad (2.38)$$

As a consequence when  $\varepsilon_c = \varepsilon_m$  we have from equations (2.34) and (2.35) that

$$f_{\text{in}}^e(z) \approx i e^{i\phi/2} \delta^{\log(r/r_\#)/\log h} h e^{-2\pi i \theta/\gamma_0} T(r_*^2/(r_0 z)) \sqrt{\frac{-W_0 \log h}{\omega \pi \varepsilon_m (\log \delta) \text{Re}[e^{i\phi/2} T(r_*^2/r_0^2)]}}, \quad (2.39)$$

which, as  $\delta \rightarrow 0$ , diverges when  $r_s < r < r_\#$  but converges to zero for  $r > r_\#$ . By similar analysis, based on equations (3.36) and (3.37) of MNMP we see that the field is resonant in the two touching annuli  $r_\# > r > \sqrt{r_c r_s}$  and  $\sqrt{r_c r_s} > r > r^\#$  where  $r^\# = \sqrt{r_c^3/r_s}$  and converges to zero outside these annuli: see figure 4.

When  $\varepsilon_c \neq \varepsilon_m$  we have from equations (2.23) and (2.25) that the resonant field in the matrix is

$$f_{\text{in}}^e(z) \approx -\eta \delta^{\log(r/r_*)/\log h} e^{-2\pi i \theta/\gamma_0} T(r_*^2/(r_0 z)) \sqrt{\frac{W_0 \log h}{\omega \pi \varepsilon_m (\log \delta) \text{Im}[\eta T(r_*^2/r_0^2)]}}, \quad (2.40)$$

which, as  $\delta \rightarrow 0$ , diverges when  $r_s < r < r_*$  but converges to zero for  $r > r_*$ . By similar analysis, based on equations (3.35) and (3.37) in MNMP, we see that the field is resonant in the entire annulus  $r_c < r < r_*$  and converges to zero outside this annulus: see figure 5.

Thus, even constant energy sources become invisible to an observer outside the cloaking region as  $\delta \rightarrow 0$ . All their energy gets trapped and absorbed in the lens. In this sense the lens behaves as a sort of ‘electromagnetic black hole’. A different sort of localization of the energy was discovered by Cui *et al.* (2005). They considered two opposing dipole sources on opposite sides of the lossless Veselago lens. Each source is positioned a distance  $d/2$  from the lens. They found that the electromagnetic energy was confined to the layer of thickness  $2d$  between the sources (i.e. the cloaking region): outside this layer the field from the nearest source cancels exactly the field from the image of the other source. In another recent development Guenneau *et al.* (2005) found that electromagnetic radiation would be trapped in the vicinity of two touching corners of negative index material.

To obtain the corresponding cloaking results for a slab rather than a coated cylinder we let  $r_s$ ,  $r_c$  and  $r_0$  tend to infinity while keeping  $d = r_s - r_c$  and  $d_0 = r_0 - r_s$  fixed. Let us define  $z = x + iy = z - r_s$  so that the polarizable line is at  $z = d_0$  and so that the slab faces will be at  $x = 0$  and  $x = -d$ . In this limit we have

$$\left. \begin{aligned} r_* &\approx r_s + d, & r_{\#} &\approx r_s + d/2, & \log h &\approx 2d/r_s, & \log(r_*/r_0) &\approx (d - d_0)/r_s, \\ \log(r_* r_s/r_0^2) &\approx (d - 2d_0)/r_s, & \log(z/r_0) &\approx (z - d_0)/r_s, \\ \frac{r_*^2}{r_0^2} &\approx 1 + \frac{2d - 2d_0}{r_s}, & \frac{r_*^2}{r_0 z} &\approx 1 + \frac{2d - d_0 - z}{r_s}. \end{aligned} \right\} \quad (2.41)$$

Also we use the approximation, given in equation (4.3) of MNMP, that  $T(1 + b/r_s) \approx r_s Q(b)$ , where

$$Q(b) \equiv \int_{-\infty}^{\infty} dv \frac{e^{vb}}{1 + e^{i\phi + 2dv}} = \frac{\pi e^{-i\phi b/(2d)}}{2d \sin[\pi b/(2d)]}. \quad (2.42)$$

For a polarizable line source with these approximations (2.27) and (2.28) reduce to

$$\left. \begin{aligned} f_{\text{in}}^e &\approx 2d(E_x + \alpha^{-1} k_0^e) \delta^{(z-d_0)/2d} Q(2d - d_0 - z)/[Q(2d - 2d_0) \log \delta], \\ f_{\text{in}}^o &\approx 2d(E_y - \alpha^{-1} k_0^o) \delta^{(z-d_0)/2d} Q(2d - d_0 - z)/[Q(2d - 2d_0) \log \delta], \end{aligned} \right\} \quad (2.43)$$

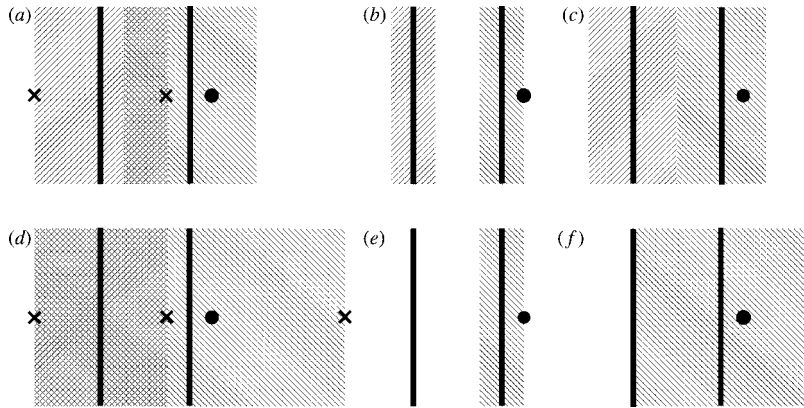


Figure 6. The resonant regions, represented by the shaded regions, for a line dipole source, represented by the solid circle, outside a slab having permittivity  $\epsilon_s$  close to  $-\epsilon_m$  and with interfaces represented by the solid lines. (a), (b) and (c) are for  $\epsilon_c = \epsilon_m$ , while (d), (e) and (f) are for  $\epsilon_c \neq \epsilon_m$ , where  $\epsilon_m$  is the permittivity on the (front) side of the slab where the source is located, while  $\epsilon_c$  is the permittivity on the other side of the slab. (a) and (d) are for a line dipole source with  $k^e$  and  $k^o$  fixed. (b) and (e) are for a polarizable line dipole source. (c) and (f) are for a constant energy source. The crosses denote ghost sources, i.e. image sources in the physical region, and the cross hatched areas are where two resonant regions overlap.

while equation (2.25) implies

$$c(\delta) \approx \eta[(1/2d)\log \delta]\delta^{(d_0-d)/d}Q(2d-2d_0) \quad \text{when } \epsilon_c \neq \epsilon_m, \quad d_0 < d, \quad (2.44)$$

and equation (2.35) reduces to

$$c(\delta) \approx -ie^{i\phi/2}[(1/2d)\log \delta]\delta^{(2d_0-d)/2d}Q(2d-2d_0) \quad \text{when } \epsilon_c = \epsilon_m, \quad d_0 < d/2. \quad (2.45)$$

For a polarizable line source with  $\epsilon_c = \epsilon_m$  and  $d_0 < d/2$  (corresponding to the symmetric lens studied by Pendry (2000)) the field is resonant in two layers each of thickness  $2d_0$ , one centered at the front interface and the other centered at the back interface. For a constant energy source with  $\epsilon_c = \epsilon_m$  and  $d_0 < d/2$  the field is resonant in two touching layers each of thickness  $d$  also centered at the interfaces. For a polarizable line source with  $\epsilon_c \neq \epsilon_m$  and  $d_0 < d$  (corresponding to the asymmetric lens studied by Ramakrishna *et al.* (2002)) the field is resonant in the layer of thickness  $2d_0$  centered at the front interface (i.e. in the region  $d_0 > x > -d_0$ ). For a constant energy source with  $\epsilon_c = \epsilon_m$  and  $d_0 < d$  the field is resonant in a layer of thickness  $2d$  extending from a distance  $d$  in front of the lens to the back interface of the lens (i.e. in the region  $d > x > -d$ ). The locations of the different resonant regions are summarized in figure 6.

### 3. A proof of cloaking for an arbitrary number of polarizable line dipoles

It is not clear if the concept of ‘effective polarizability’ has much use when two or more polarizable lines are positioned in the cloaking region since each polarizable line will also interact with the resonant regions generated by the other polarizable lines and if the polarizable lines are not all on a plane containing the coated cylinder axis then these interactions will oscillate as  $\delta \rightarrow 0$ . However,

we will see here that nevertheless the dipole moment of each polarizable line in the cloaking region must go to zero as  $\delta \rightarrow 0$  and in such a way that no resonant field extends outside the cloaking region. This is not too surprising. Based on the results for a single dipole line we expect that a resonant field extending outside the cloaking region would cost infinite energy, and the only way to avoid this is for the dipole moment of each polarizable line in the cloaking region to go to zero as  $\delta \rightarrow 0$ .

Here we limit our attention to the cylindrical lens with the core having (approximately) the same permittivity as the matrix. Also to simplify the analysis we assume the core (but not the matrix) has some small loss. Specifically we assume

$$\varepsilon_m = 1, \quad \varepsilon_s = -1 + i\varepsilon_s, \quad \varepsilon_c = 1 + i\varepsilon_c, \quad (3.1)$$

with  $\varepsilon_s$  and  $\varepsilon_c$  having positive real parts and approaching zero in such a way that the ratio  $\gamma = \varepsilon_c/\varepsilon_s$ , which could be complex, remains fixed and  $\phi$  given by equation (2.1) also remains fixed. In this limit (2.1) implies  $(\varepsilon_s + \varepsilon_c)\varepsilon_s \approx 4\delta e^{i\phi}$  and since  $\varepsilon_s$  and  $\varepsilon_c$  have positive real parts we deduce that  $\phi$  is not equal to  $\pi$  or  $-\pi$ . Using the relation  $\gamma = \varepsilon_c/\varepsilon_s$  we see that

$$\varepsilon_s = 2\sqrt{\delta}e^{i\phi/2}/\sqrt{1+\gamma}, \quad \varepsilon_c = 2\sqrt{\delta}e^{i\phi/2}\gamma/\sqrt{1+\gamma}. \quad (3.2)$$

The potential in the core due to a single dipole at  $z_0$ , with  $|z_0| > r_s$  is

$$V_c(r, \theta, z_0) = [f_c^e(z, z_0) + f_c^e(\bar{z}, \bar{z}_0)]/2 + [f_c^o(z, z_0) - f_c^o(\bar{z}, \bar{z}_0)]/(2i), \quad (3.3)$$

where  $z = re^{i\theta}$ ,  $z_0 = r_0e^{i\theta_0}$ , and

$$f_c^p(z, z_0) = \sum_{\ell=0}^{\infty} E_{\ell}^p(z_0) z^{\ell}, \quad (3.4)$$

for  $p=e$  and  $p=o$ , and for all  $\ell \neq 0$

$$E_{\ell}^p(z_0) = \frac{-k^p \beta(\delta) (h/z_0)^{\ell}}{r_0(1 + \delta e^{i\phi} h^{\ell})}, \quad \beta(\delta) \equiv \frac{4\varepsilon_s}{(\varepsilon_s - \varepsilon_c)(1 - \varepsilon_s)}, \quad (3.5)$$

and  $\beta(\delta)$  depends on  $\delta$  through the dependence of  $\varepsilon_s$  and  $\varepsilon_c$  on  $\delta$  but tends to 1 as  $\delta \rightarrow 0$ . Here  $k^e$  and  $k^o$  are the amplitudes of the dipole components which have even and odd symmetry about the line  $\theta = \theta_0$ , respectively. These formulae agree with the formulae given in equations (2.5) and (3.13) of MNMP when  $z_0 = r_0$  is real, and since the rotational invariance property  $V_c(r, \theta, z_0) = V_c(r, \theta - \theta_0, r_0)$  is satisfied we deduce that the formula is correct when  $z_0$  is complex. (In the formula for  $E_{\ell}^p(z_0)$  the requirement of rotational invariance necessitates the factor of  $1/(r_0 z_0^{\ell})$ , rather than say a factor of  $1/z_0^{\ell+1}$ ).

If there are  $m$  dipoles at  $z_1, z_2, \dots, z_m$  (where  $z_i \neq z_j$  for all  $i \neq j$ ) all outside the coated cylinder then, by the superposition principle, the potential in the core is

$$V_c = \sum_{\ell=0}^{\infty} E_{\ell}^{(1)} z^{\ell} + E_{\ell}^{(2)} \bar{z}^{\ell}, \quad (3.6)$$

where for  $\ell \neq 0$

$$\begin{aligned} E_\ell^{(1)} &= \frac{h^\ell \beta(\delta)}{(1 + \delta e^{i\phi} h^\ell)} \sum_{j=1}^m (k_j^{(1)}/r_j)(1/z_j)^\ell, \\ E_\ell^{(2)} &= \frac{h^\ell \beta(\delta)}{(1 + \delta e^{i\phi} h^\ell)} \sum_{j=1}^m (k_j^{(2)}/r_j)(1/\bar{z}_j)^\ell, \end{aligned} \quad (3.7)$$

in which

$$k_j^{(1)} = (-k_j^e + i k_j^o)/2, \quad k_j^{(2)} = -(k_j^e + i k_j^o)/2. \quad (3.8)$$

Let us suppose the dipoles positioned at  $z_1, z_2, \dots, z_g$  with  $1 \leq g \leq m$  are in the cloaking region, while the remainder of the dipoles are outside the cloaking region, i.e.

$$|z_j| \leq r_\# \quad \text{for all } j \leq g, \quad |z_j| > r_\# \quad \text{for all } j > g, \quad (3.9)$$

where we allow for the special case where some of the dipoles have  $|z_j| = r_\#$ : as we will see, these are also cloaked. We do not specify how the set of dipole moments  $\{k_1, k_2, \dots, k_m\}$  depends on  $\delta$  except for the following.

- (i) We assume that each dipole outside the cloaking region has moments which converge to fixed limits as  $\delta \rightarrow 0$

$$\lim_{\delta \rightarrow 0} (k_j^e(\delta), k_j^o(\delta)) = (k_{j0}^e, k_{j0}^o) \quad \text{for all } j > g. \quad (3.10)$$

The dipole moments  $k_j^e(\delta)$  and  $k_j^o(\delta)$  inside or outside the cloaking region are assumed to depend linearly on the field acting upon them, since nonlinearities would generate higher order frequency harmonics. Some of them could be energy sinks, although at least one of them should be an energy source.

- (ii) We assume that the energy absorbed per unit time per unit length of the coated cylinder remains bounded as  $\delta \rightarrow 0$ , as, e.g. must be the case if the line sources only supply a finite amount of energy per unit time per unit length. We let  $W_{\max}$  be the maximum amount of energy available per unit time per unit length. It is supposed that the quasistatic limit is being taken not by letting the frequency  $\omega$  tend to zero, but instead by fixing the frequency  $\omega$  and reducing the spatial size of the system and using a coordinate system which is appropriately rescaled.

We need to show that, because the energy absorption in the core remains bounded, the dipole moments in the cloaking region go to zero as  $\delta \rightarrow 0$  and the resonant field does not extend outside the cloaking region,  $r \leq r_\#$ . This is certainly true when only one polarizable line is present but as cancellation effects can occur (the energy absorption associated with two line dipoles can be less than the absorption associated with either line dipole acting separately) a proof is needed.

To do this, we bound  $k_i^e$  and  $k_i^o$  for any given  $i \leq g$  using the fact that the energy loss within the lens is bounded by  $W_{\max}$ . If  $W_c = W_c(\delta)$  represents

the energy dissipated in the core, then we have the inequality

$$W_c = (\omega/2)\varepsilon_c'' \int_0^{r_c} r \, dr \int_0^{2\pi} d\theta \mathbf{E}(z) \cdot \overline{\mathbf{E}(z)} \geq (\omega/2)\varepsilon_c'' \int_0^{r_c} r \, dr \times \int_0^{2\pi} d\theta E_x(z) \overline{E_x(z)}, \quad (3.11)$$

in which  $\varepsilon_c'' = \text{Im}(\varepsilon_c) = \text{Re}(\epsilon_c)$  and  $E_x(z)$  is the  $x$  component of the electric field in the core given by

$$E_x(z) = -\frac{\partial V_c}{\partial x} = -\sum_{\ell=1}^{\infty} \ell r^{\ell-1} (E_{\ell}^{(1)} e^{i(\ell-1)\theta} + E_{\ell}^{(2)} e^{-i(\ell-1)\theta}). \quad (3.12)$$

Substituting this expression for the electric field back in equation (3.11) and using the orthogonality properties of Fourier modes we then have

$$\begin{aligned} 2W_c/\omega &\geq 2\pi\varepsilon_c'' \int_0^{r_c} dr (E_1^{(1)} + E_1^{(2)}) \overline{(E_1^{(1)} + E_1^{(2)})} r + \sum_{\ell=2}^{\infty} \ell^2 r^{2\ell-1} (E_{\ell}^{(1)} \overline{E_{\ell}^{(1)}} + E_{\ell}^{(2)} \overline{E_{\ell}^{(2)}}) \\ &\geq \pi\varepsilon_c'' r_c^2 (E_1^{(1)} + E_1^{(2)}) \overline{(E_1^{(1)} + E_1^{(2)})} + \pi\varepsilon_c'' \sum_{\ell=2}^{\infty} \ell r_c^{2\ell} (E_{\ell}^{(1)} \overline{E_{\ell}^{(1)}} + E_{\ell}^{(2)} \overline{E_{\ell}^{(2)}}) \\ &\geq \pi\varepsilon_c'' \sum_{\ell=n}^{n+m-1} \ell r_c^{2\ell} (E_{\ell}^{(1)} \overline{E_{\ell}^{(1)}} + E_{\ell}^{(2)} \overline{E_{\ell}^{(2)}}) = \pi\varepsilon_c'' \sum_{k=0}^{m-1} b_k (U_k \overline{U_k} + V_k \overline{V_k}), \end{aligned} \quad (3.13)$$

where the last identity is obtained using equation (3.7) with the definitions

$$\left. \begin{aligned} b_k &\equiv (n+k)(r_*/r_i)^{2n+2k} |\beta(\delta)/(1+\delta h^{n+k} e^{i\phi})|^2, \\ U_k &\equiv \sum_{j=1}^m u_j (r_i/z_j)^k, \quad u_j \equiv (r_i/z_j)^n k_j^{(1)}/r_j, \\ V_k &\equiv \sum_{j=1}^m v_j (r_i/\bar{z}_j)^k, \quad v_j \equiv (r_i/\bar{z}_j)^n /k_j^{(2)}/r_j, \end{aligned} \right\} \quad (3.14)$$

in which  $k = \ell - n$ ,  $n \geq 2$  remains to be chosen, and  $i \leq g$ . From equation (3.14) it follows that  $\mathbf{U} = \mathbf{M}\mathbf{u}$  and  $\mathbf{V} = \mathbf{M}\mathbf{v}$ , where  $\mathbf{M}$  is the Vandermonde matrix

$$M = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ r_i/z_1 & r_i/z_2 & r_i/z_3 & \dots & r_i/z_m \\ (r_i/z_1)^2 & (r_i/z_2)^2 & (r_i/z_3)^2 & \dots & (r_i/z_m)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (r_i/z_1)^{m-1} & (r_i/z_2)^{m-1} & (r_i/z_3)^{m-1} & \dots & (r_i/z_m)^{m-1} \end{pmatrix}. \quad (3.15)$$

From the well-known formula for the determinant of a Vandermonde matrix it follows that  $\mathbf{M}$  is non-singular. Therefore there exists a constant  $c_i > 0$  (which is the reciprocal of the norm of  $\mathbf{M}^{-1}$  and which only depends on  $i$ ,  $m$  and the  $z_j$ )

such that  $|\mathbf{U}| \geq c_i |\mathbf{u}|$  and  $|\mathbf{V}| \geq c_i |\mathbf{v}|$ , implying

$$|\mathbf{U}|^2 + |\mathbf{V}|^2 \geq c_i^2 (|\mathbf{u}|^2 + |\mathbf{v}|^2) \geq c_i^2 (|u_i|^2 + |v_i|^2) = c_i^2 (|k_i^{(1)}|^2 + |k_i^{(2)}|^2) / r_i^2. \quad (3.16)$$

Next we need to select  $n$  and find a lower bound on  $b_k$  which is independent of  $k$ . Let  $s = -\log \delta / \log h$  (so  $\delta h^s = 1$ ) and take  $n$  as the smallest integer greater than or equal to  $s$  so  $n \geq s \geq n-1$ . Then since  $r_* > r_i$  we have

$$(r_*/r_i)^{2n} \geq (r_*/r_i)^{2s} = \delta^{-2 \log(r_*/r_i) / \log h} = \delta^{-1/2} \delta^{-2 \log(r_*/r_i) / \log h}. \quad (3.17)$$

Also the following inequalities hold for  $m-1 \geq k \geq 0$

$$1 = \delta h^s \leq \delta h^n \leq \delta h^{n+k} \quad \text{and} \quad \delta h^{n+k} \leq \delta h^{s+1+k} \leq \delta h^{s+m} = h^m. \quad (3.18)$$

So it follows that

$$|1 + \delta h^{n+k} e^{i\phi}| \leq a \equiv \max_{1 \leq t \leq h^m} |1 + t e^{i\phi}|, \quad (3.19)$$

and  $a$  is independent of  $\delta$ . From the bounds (3.17) and (3.19) we deduce that

$$\begin{aligned} b_k &\geq s(r_*/r_i)^{2k} \delta^{-1/2} \delta^{-2 \log(r_*/r_i) / \log h} |\beta(\delta)|^2 / a^2 \\ &\geq -(\log \delta / \log h) \delta^{-1/2} \delta^{-2 \log(r_*/r_i) / \log h} |\beta(\delta)|^2 / a^2. \end{aligned} \quad (3.20)$$

Combining inequalities gives

$$\begin{aligned} 2W_c / \omega &\geq \frac{\pi \varepsilon_c'' |\beta(\delta)|^2}{\sqrt{\delta} a^2 \log h} (-\log \delta) \delta^{-2 \log(r_*/r_i) / \log h} (|\mathbf{U}|^2 + |\mathbf{V}|^2) \\ &\geq \frac{\pi \varepsilon_c'' |\beta(\delta)|^2 c_i^2}{\sqrt{\delta} a^2 r_i^2 \log h} (-\log \delta) \delta^{-2 \log(r_*/r_i) / \log h} (|k_i^{(1)}|^2 + |k_i^{(2)}|^2), \end{aligned} \quad (3.21)$$

in which the real positive prefactor has the property that

$$\rho_i \equiv \lim_{\delta \rightarrow 0} \frac{\pi \varepsilon_c'' |\beta(\delta)|^2 c_i^2}{\sqrt{\delta} a^2 r_i^2 \log h} = \frac{2\pi c_i^2}{a^2 r_i^2 \log h} \operatorname{Re}(e^{i\phi/2} \gamma / \sqrt{1+\gamma}), \quad (3.22)$$

is positive and non-zero, where  $\operatorname{Re}(w)$  denotes the real part of  $w$ . So there exists a  $\delta_0$  such that, for all positive  $\delta < \delta_0$  and all  $i \leq g$ ,

$$\frac{\pi \varepsilon_c'' |\beta(\delta)|^2 c_i^2}{\sqrt{\delta} a^2 r_i^2 \log h} \geq \rho/2, \quad \text{where } \rho \equiv \min_{i \leq g} \rho_i > 0. \quad (3.23)$$

By the triangle inequality we have

$$|k_i^{(1)}|^2 + |k_i^{(2)}|^2 = |k_i^{(1)}|^2 + |-k_i^{(2)}|^2 \geq \max\{|k_i^e|^2, |k_i^o|^2\}, \quad (3.24)$$

and so we conclude that

$$|k_i^p| \leq 2\delta^{\log(r_*/r_i) / \log h} \sqrt{-W_c / (\omega \rho \log \delta)}, \quad (3.25)$$

which forces the dipole moment  $k_i^p$  to go to zero as  $\delta \rightarrow 0$  (even when  $r_i = r_*$ ) because  $W_c = W_c(\delta) \leq W_{\max}$ .

Now the superposition principle implies that the potential at any point  $z$  in the matrix is

$$V(z) = \sum_{j=1}^m k_j^e V_j^e(z) + k_j^o V_j^o(z), \quad (3.26)$$

where  $V_j^e(z)$  (or  $V_j^o(z)$ ) is the potential in the matrix due to an isolated line dipole at the point  $z_j$  with  $k_j^e = 1$ ,  $k_j^o = 0$  (respectively with  $k_j^e = 0$ ,  $k_j^o = 1$ ). Now according to theorem 3.2 in MNMP (which is easily extended to the case treated here where  $\varepsilon_c$  depends on  $\delta$ ) it follows that for  $|z| > \max\{r_s, r_{\#}^2/r_j\}$ ,

$$\lim_{\delta \rightarrow 0} V_j^p(z) = \tilde{V}_j^p(z) = [\tilde{f}_j^e(z) + \tilde{f}_j^e(\bar{z})]/2 + [\tilde{f}_j^o(z) - \tilde{f}_j^o(\bar{z})]/(2i), \quad (3.27)$$

where, because  $\varepsilon_c$  approaches  $\varepsilon_m$ ,

$$\tilde{f}_j^e(z) = [1/(z - z_j) + 1/(\bar{z} - \bar{z}_j)]/2, \quad \tilde{f}_j^o(z) = [1/(z - z_j) - 1/(\bar{z} - \bar{z}_j)]/(2i). \quad (3.28)$$

Also as shown above equation (3.27) in MNMP if  $r_{\#}^2/r_j > |z| > r_s$ , then  $V_j^p(z)$  diverges as  $\delta^{-a}$  where  $a = \log(r_s r_{\#}/r_j |z|)/\log h$ . If  $z_j$  is outside the cloaking region (i.e.  $j > g$ ) then  $r_{\#}^2/r_j$  will be less than  $r_{\#}$ . So using the well-known fact that

$$\lim_{\delta \rightarrow 0} e(\delta)f(\delta) = e_0 f_0, \quad \text{where } e_0 = \lim_{\delta \rightarrow 0} e(\delta), \quad f_0 = \lim_{\delta \rightarrow 0} f(\delta), \quad (3.29)$$

it follows that

$$\lim_{\delta \rightarrow 0} k_j^p V_j^p(z) = k_{j0}^p \tilde{V}_j^p(z) \quad \text{for all } |z| > r_{\#}, \quad j > h, \quad p = e, o. \quad (3.30)$$

If  $z_i$  is inside the cloaking region (i.e.  $i \leq g$ ) and  $|z| > r_{\#}^2/r_i$  then equations (3.27), (3.29) and the fact that  $|k_i^p|$  tends to zero implies that  $k_i^p V_i^p(z)$  will tend to zero. For  $r_{\#}^2/r_i > |z| > r_{\#}$  we have that  $V_i^p(z)$  scales as  $\delta^{-a}$  with  $a = \log(r_s r_{\#}/(r_i |z|))/\log h$  while from equation (3.25)  $k_i^p$  scales at worst as  $\delta^b/(-\log \delta)$  with  $b = \log(r_{\#}/r_i)/\log h$ . So their product  $k_i^p V_i^p(z)$  will scale at worst as  $\delta^{b-a}/(-\log \delta)$  where  $b - a = \log(|z|/r_{\#})/\log h$ . This goes to zero as  $\delta \rightarrow 0$  when  $|z| > r_{\#}$ . By taking the limit  $\delta \rightarrow 0$  of both sides of equation (3.26) we conclude that

$$\lim_{\delta \rightarrow 0} V(z) = \sum_{j=h+1}^m k_{j0}^e \tilde{V}_j^e(z) + k_{j0}^o \tilde{V}_j^o(z) \quad \text{for all } |z| > r_{\#}, \quad (3.31)$$

which proves that the coated cylinder and all the line dipoles inside the cloaking region are invisible outside the cloaking region in this limit.

More can be said if there is only one dipole line outside the cloaking region, i.e.  $g = m - 1$ , and the dipoles inside the cloaking region are always quasistatic energy sinks, in the sense that for all  $j \leq g$  the inequality

$$\text{Im}[\bar{k}_j^e E_x^{0(j)} - \bar{k}_j^o E_y^{0(j)}] \leq 0, \quad (3.32)$$

is satisfied no matter what is the value of the field  $(E_x^{0(j)}, E_y^{0(j)})$  acting on the line dipole at  $z_j$ . For example, if the line dipole at  $x_j$  responds linearly to the local field with

$$\begin{pmatrix} k_j^e \\ -k_j^o \end{pmatrix} = \alpha_j \begin{pmatrix} E_x^{0(j)} \\ E_y^{0(j)} \end{pmatrix} \quad \text{for all } j \leq g, \quad (3.33)$$

then equation (3.32) will be satisfied provided  $\alpha_j + \alpha_j^T$  has a positive semidefinite imaginary part.

From equation (4.10) of MNMP, generalized to allow for more than one line dipole, it follows that

$$\begin{aligned} W_c &\leq (\omega/2) \int_{|z| \leq r_s} dx dy \varepsilon'' \mathbf{E}(x, y) \cdot \overline{\mathbf{E}(x, y)} = \omega\pi \sum_{j=1}^m \text{Im}[\bar{k}_j^e E_x^{0(j)} - \bar{k}_j^o E_y^{0(j)}] \\ &\leq \omega\pi \text{Im}[\bar{k}_m^e E_x^{0(m)} - \bar{k}_m^o E_y^{0(m)}] \leq \omega\pi |\bar{k}_m^e E_x^{0(m)} - \bar{k}_m^o E_y^{0(m)}| \\ &\leq \omega\pi [|k_m^e| |E_x^{0(m)}| + |k_m^o| |E_y^{0(m)}|]. \end{aligned} \quad (3.34)$$

Also equation (3.26) implies

$$E_x^{0(m)} = \sum_{j=1}^g k_j^e E_x^{e(j)}(z_m) + k_j^o E_x^{o(j)}(z_m), \quad E_y^{0(m)} = \sum_{j=1}^g k_j^e E_y^{e(j)}(z_m) + k_j^o E_y^{o(j)}(z_m), \quad (3.35)$$

where

$$E_x^{p(j)}(z) \equiv -\frac{\partial V_j^p(z)}{\partial x}, \quad E_y^{p(j)}(z) \equiv -\frac{\partial V_j^p(z)}{\partial y}. \quad (3.36)$$

So we have

$$\left. \begin{aligned} |E_x^{0(m)}| &\leq \sum_{j=1}^g |k_j^e| |E_x^{e(j)}(z_m)| + |k_j^o| |E_x^{o(j)}(z_m)|, \\ |E_y^{0(m)}| &\leq \sum_{j=1}^g |k_j^e| |E_y^{e(j)}(z_m)| + |k_j^o| |E_y^{o(j)}(z_m)|. \end{aligned} \right\} \quad (3.37)$$

Now from equation (3.25) when  $\delta < \delta_0$

$$|k_i^p| \leq 2\delta^{\log(r_{\#}/r_{\max})/\log h} \sqrt{-W_c/(\omega\rho \log \delta)}, \quad \text{where } r_{\max} \equiv \max_{i \leq g} r_i, \quad (3.38)$$

and this with the inequalities (3.34) and (3.37) implies

$$W_c \leq \psi \delta^{\log(r_{\#}/r_{\max})/\log h} \sqrt{-\omega W_c / \log \delta}, \quad (3.39)$$

where

$$\psi = 2\pi \sqrt{(1/\rho)} \sum_{j=1}^g |k_m^e| (|E_x^{e(j)}(z_m)| + |E_x^{o(j)}(z_m)|) + |k_m^o| (|E_y^{e(j)}(z_m)| + |E_y^{o(j)}(z_m)|). \quad (3.40)$$

As  $\delta \rightarrow 0$  this tends to

$$\psi_0 = 2\pi \sqrt{(1/\rho)} \sum_{j=1}^g |k_m^e| (|\tilde{E}_x^{e(j)}(z_m)| + |\tilde{E}_x^{o(j)}(z_m)|) + |k_m^o| (|\tilde{E}_y^{e(j)}(z_m)| + |\tilde{E}_y^{o(j)}(z_m)|), \quad (3.41)$$

where

$$\tilde{E}_x^{\text{p}(j)}(z_m) = -\frac{\partial \tilde{V}_j^{\text{p}}(z)}{\partial x}, \quad \tilde{E}_y^{\text{p}(j)}(z_m) = -\frac{\partial \tilde{V}_j^{\text{p}}(z)}{\partial y}. \quad (3.42)$$

So there exists a positive  $\delta_1 < \delta_0$  such that  $\psi \leq 2\psi_0$  for all  $\delta < \delta_1$  and from equation (3.39) we deduce that

$$W_c \leq -4\omega\psi_0^2\delta^2 \log(r_{\#}/r_{\max})/\log(h)/\log \delta, \quad (3.43)$$

which goes to zero as  $\delta \rightarrow 0$ . Similarly the energy absorption in the shell goes to zero as  $\delta \rightarrow 0$ . Combining this with equation (3.25) gives an improved bound on the  $i$ th dipole moment in the cloaking region:

$$|k_i^{\text{p}}| \leq -4\psi_0 \sqrt{(1/\rho)} \delta^{\log[r_{\#}^2/(r_{\max}r_i)]/\log h}/\log \delta. \quad (3.44)$$

For  $r_{\#}^2/r_j > |z| > r_{\max}$  we have that  $V_j^{\text{p}}(z)$  scales as  $\delta^{-a}$  with  $a = \log(r_{\#}^2/(r_j|z|))/\log h$  while  $k_j^{\text{p}}$  scales at worst as  $\delta^b/(-\log \delta)$  with  $b = \log[r_{\#}^2/(r_j r_{\max})]/\log h$ . So their product  $k_j^{\text{p}} V_j^{\text{p}}(z)$  will scale at worst as  $\delta^{b-a}/(-\log \delta)$  where  $b-a = \log(|z|/r_{\max})/\log h$ . This goes to zero as  $\delta \rightarrow 0$  when  $|z| > r_{\max}$ . So all the dipoles in the cloaking region will have vanishingly small contribution to the potential  $V(z)$  outside the radius  $r_{\max}$ . There will be a resonant field in the region between  $r_{\max}$  and  $r_{\#}$  if and only if  $r_{\#}^2/r_m > r_{\max}$  and even if this resonant field is present, its asymptotic form will not be influenced by the dipoles in the cloaking region.

By the superposition principle this last result extends to the case where an arbitrary number of line dipoles lie outside the cloaking region provided their moments  $(k_j^e, k_j^o)$  for  $j > g$  do not depend on  $\delta$  and provided the line dipoles inside the cloaking region have a linear response of the form (3.33) with the imaginary part of  $\alpha_j + \alpha_j^T$  being positive semidefinite for all  $j \leq g$ .

#### 4. Cloaking properties of the Veselago slab lens

Let us now move away from quasistatics and investigate the cloaking properties of the Veselago slab lens at fixed but arbitrary frequency  $\omega$ . We assume the lens has relative permittivity  $\epsilon_s = -1 + i\epsilon$  and relative permeability  $\mu_s = -1 + i\nu$ , where  $\epsilon$  and  $\nu$  are now assumed to be real, and that the surrounding medium has relative permittivity and relative permeability both equal to 1.

We assume that the source is a line electrical dipole positioned along the  $Z$ -axis,  $(x, y) = (0, 0)$  with the slab faces at the planes  $x = d_0$  and  $x = d_0 + d$ , with  $d$  being the slab thickness and  $d_0$  being the distance from the source to the lens. For TM polarization all the electromagnetic field components are easily calculated once one has determined the only non-zero component of the magnetic field  $H_Z(x, y)$ , where we have used a capital  $Z$  for the  $z$ -coordinate to avoid confusion with  $z = x + iy$ . By the superposition principle  $H_Z(x, y)$  is given by the expression

$$H_Z(x, y) = \int_{-\infty}^{\infty} dk_y a(k_y) \tau(x, y; k_y), \quad (4.1)$$

where for a line dipole source

$$a(k_y) = -\omega[k^e(k_y/k_x) + ik^o]/2, \quad \text{with } k_x = \sqrt{\omega^2/c^2 - k_y^2}, \quad (4.2)$$

in which  $k^e$  is the (possibly complex) strength of the dipole component which has an electric field component  $E_x(x, y)$  with even symmetry about the  $x$ -axis (i.e. with  $E_x(x, -y) = E_x(x, y)$  and  $H_Z(x, -y) = -H_Z(x, y)$ ) and  $k^o$  is the (possibly complex) strength of the dipole component which has  $E_x(x, y)$  with odd symmetry about the  $x$ -axis (i.e. with  $E_x(x, -y) = -E_x(x, y)$  and  $H_Z(x, -y) = H_Z(x, y)$ ): these have been normalized so that they are consistent with the quasistatic definitions of  $k^e$  and  $k^o$  (which are not to be confused with wavevectors such as  $k_x$  and  $k_y$  which have subscripts). The transfer function  $\tau(x, y; k_y)$  represents the solution for  $H_Z$  when a plane wave with an incident field  $H_Z^{\text{inc}} = e^{i(k_x x + k_y y - \omega t)}$  comes towards the lens from the left. Let  $\tau_m(x, y; k_y)$ ,  $\tau_s(x, y; k_y)$  and  $\tau_c(x, y; k_y)$  denote the expressions for  $\tau(x, y; k_y)$  in front (to the left) of the slab lens, in the slab lens, and behind (to the right) of the slab lens, respectively. In each region  $\tau(x, y; k_y)$  is a linear combination of two plane waves except behind the slab lens, where there is only an outgoing plane wave. The coefficients can be determined from the requirement of continuity of the tangential components of the magnetic and electric fields across each interface, i.e. from the continuity of  $\tau(x, y; k_y)$  and  $(1/\epsilon)\partial\tau(x, y; k_y)/\partial x$ . In this way explicit expressions for these transfer functions can be derived (e.g. Kong (2002) and Podolskiy & Narimanov (2005)) but here we will only need their asymptotic forms.

For fixed  $k_y$  we have

$$\begin{aligned} \lim_{\epsilon, \nu \rightarrow 0} \tau(x, y; k_y) &= \tau^0(x, y; k_y) \equiv e^{i(k_x x + k_y y)} & \text{for } x < d_0, \\ &\equiv e^{i[k_x(2d_0 - x) + k_y y]} & \text{for } d_0 < x < d + d_0, \\ &\equiv e^{i[k_x(x - 2d) + k_y y]} & \text{for } x > d + d_0. \end{aligned} \quad (4.3)$$

Let us choose a very large positive number  $k_c$  which is to remain fixed as  $\epsilon, \nu \rightarrow 0$ . Then the integral (4.1) can be rewritten as

$$H_Z(x, y) = [A(x, y) + B(x, y) + C(x, y)], \quad (4.4)$$

where

$$\left. \begin{aligned} A(x, y) &= \int_{k_c}^{\infty} dk_y a(k_y) \tau(x, y; k_y), \\ B(x, y) &= \int_{-k_c}^{k_c} dk_y a(k_y) \tau(x, y; k_y), \\ C(x, y) &= \int_{-\infty}^{-k_c} dk_y a(k_y) \tau(x, y; k_y), \end{aligned} \right\} \quad (4.5)$$

and let us define

$$H_Z^0(x, y) = \lim_{\epsilon, \nu \rightarrow 0} B(x, y) = \int_{-k_c}^{k_c} dk_y a(k_y) \tau^0(x, y; k_y). \quad (4.6)$$

It follows from equation (4.3) that this field has the mirroring properties

$$\left. \begin{aligned} H_Z^0(x, y) &= H_Z^0(2d_0 - x, y) & \text{for } 0 < x < 2d_0, \\ H_Z^0(x, y) &= H_Z^0(2d_0 + 2d - x, y) & \text{for } d_0 < x < d_0 + 2d, \end{aligned} \right\} \quad (4.7)$$

which combine to give the shifting property

$$H_Z^0(x, y) = H_Z^0(x - 2d, y) \quad \text{for } 2d < x < d_0 + 2d, \quad (4.8)$$

that is responsible for the superlensing.

In the region  $0 < x < d_0$  the field  $H_Z^0(x, y)$  is approximately that due to the dipole line with the lens absent, except near the plane  $x=0$ . Incidentally, the analytic continuation of this field to the region  $x < 0$  will be an enormously large field with spatial oscillations on the length scale of  $1/k_c$ . In the region  $d_0 < x < 2d_0$  the field  $H_Z^0(x, y)$  is approximately that due to a solitary ghost line dipole at  $(x, y) = (2d_0, 0)$ , except near the plane  $x = 2d_0$ . Similarly, in the region  $2d < x < d_0 + 2d$  the field  $H_Z^0(x, y)$  is approximately that due to a solitary ghost line dipole at  $(x, y) = (2d, 0)$ , except near the plane  $x = 2d$ . In the region  $2d_0 < x < 2d$  the field  $H_Z^0(x, y)$  will be enormously large (but bounded for fixed  $k_c$ ) with spatial oscillations on the length scale of  $1/k_c$ . In this region and in the region  $2d_0 - d < x < d$  the field  $H_Z^0(x, y)$  will be dwarfed by the field  $A(x, y) + C(x, y)$  for sufficiently small  $\delta$ .

For large  $|k_y|$  and small loss (i.e. small  $\epsilon$  and  $\nu$ ) very good approximations to the transfer functions have been derived by Podolskiy & Narimanov (2005) and Podolskiy *et al.* (2005) and are given by

$$\left. \begin{aligned} \tau_m(x, y; k_y) &\approx e^{-\kappa x + i k_y y} + \frac{i \xi e^{\kappa(2d-2d_0+x)}}{1 + \xi^2 e^{2\kappa d}} e^{i k_y y}, \\ \tau_s(x, y; k_y) &\approx \frac{e^{\kappa(x-2d_0)} + i \xi e^{\kappa(2d-x)}}{(1 + i \xi)(1 + \xi^2 e^{2\kappa d})} e^{i k_y y}, \\ \tau_c(x, y; k_y) &\approx \frac{e^{\kappa(2d-x)}}{1 + \xi^2 e^{2\kappa d}} e^{i k_y y}, \end{aligned} \right\} \quad (4.9)$$

in which  $\kappa = \sqrt{k_y^2 - \omega^2/c^2}$  and  $\xi$  is the loss function

$$\xi = \frac{1}{2} \left[ \epsilon + \frac{\epsilon + \nu}{2(k_y^2 c^2 / \omega^2 - 1)} \right], \quad (4.10)$$

where the first expression has been kindly supplied to us by Viktor Podolskiy (2005, personal communication). For very large  $|k_y|$  and very small loss, we have that  $\xi \approx \epsilon/2$  and  $\kappa \approx |k_y|$  so the approximate expressions for the transfer functions reduce to

$$\left. \begin{aligned} \tau_m(x, y; k_y) &\approx e^{-|k_y|x + i k_y y} + \frac{i(\epsilon/2) e^{|k_y|(2d-2d_0+x)}}{1 + (\epsilon/2)^2 e^{2|k_y|d}} e^{i k_y y}, \\ \tau_s(x, y; k_y) &\approx \frac{e^{|k_y|(x-2d_0)} + i(\epsilon/2) e^{|k_y|(2d-x)}}{1 + (\epsilon/2)^2 e^{2|k_y|d}} e^{i k_y y}, \\ \tau_c(x, y; k_y) &\approx \frac{e^{|k_y|(2d-x)}}{1 + (\epsilon/2)^2 e^{2|k_y|d}} e^{i k_y y}, \end{aligned} \right\} \quad (4.11)$$

and in this limit

$$a(k_y) \approx -\omega[k^e(k_y/|k_y|) + i k^o]/2. \quad (4.12)$$

The important observation is that these asymptotic expressions (with the exception of the scale factor of  $\omega$  in equation (4.12)) are independent of the frequency  $\omega$ . Since whether or not the integrals  $A(x, y)$  and  $C(x, y)$  converge or diverge as  $\epsilon$  and  $\nu$  tend to zero is determined by the asymptotic form of the transfer functions we conclude that the resonant regions at any frequency must be located in the same areas as in the quasistatic limit, i.e. in two layers of equal thickness, one centered at the front interface of the lens and the other centered at the back interface. Furthermore the asymptotic expressions for the fields in the resonant regions should be the same expressions as those in the quasistatic limit, given by equations (4.6)–(4.9) of MNMP, and as a result the effective polarizability should be the same as in the quasistatic case.

Let us now directly see this. We need to estimate integrals of the form

$$I(b) \equiv \int_{k_c}^{\infty} dk_y \frac{e^{k_y b}}{1 + (\epsilon/2)^2 e^{2dk_y}} = \int_{-\infty}^{-k_c} dk_y \frac{e^{-k_y b}}{1 + (\epsilon/2)^2 e^{-2dk_y}}, \quad (4.13)$$

for complex values of  $b = b' + ib''$  in the limit as  $\epsilon \rightarrow 0$ . Clearly if  $b'$  is negative we have the estimate

$$|I(b)| \leq \int_{k_c}^{\infty} dk_y |e^{k_y b}| = \int_{k_c}^{\infty} dk_y e^{k_y b'} = -e^{b' k_c} / b', \quad (4.14)$$

and since  $k_c$  is large the integral is negligibly small except when  $b'$  is very small. When  $b' > 0$  let the transition point  $k_t$  be defined by  $(\epsilon/2)e^{dk_t} = 1$ , i.e.  $k_t = -(1/d)\log(\epsilon/2)$ , and let us change the variable of integration from  $k_y$  to  $v = k_y - k_t$ . Then the integral becomes

$$I(b) = e^{k_t b} \int_{k_c - k_t}^{\infty} dv \frac{e^{vb}}{1 + e^{2dv}} \approx e^{k_t b} \int_{-\infty}^{\infty} dv \frac{e^{vb}}{1 + e^{2dv}} = (\epsilon/2)^{-b/d} Q_0(b), \quad (4.15)$$

where  $Q_0(b)$  is obtained by setting  $\phi=0$  in the integral (2.42) giving

$$Q_0(b) = \frac{\pi}{2d \sin[\pi b/(2d)]}. \quad (4.16)$$

In making the approximation (4.15) we have assumed that  $\epsilon$  is so incredibly small that  $k_t = -(1/d)\log(\epsilon/2) \gg k_c$ . From equation (4.15) we see that  $(\epsilon/2)I(b) \approx (\epsilon/2)^{(d-b)/d} Q_0(b)$  and a quantity like this is negligible in the limit  $\epsilon \rightarrow 0$  when  $b' < d$ .

Let  $A_m(x, y)$ ,  $A_s(x, y)$ ,  $A_c(x, y)$  and  $B_m(x, y)$ ,  $B_s(x, y)$ ,  $B_c(x, y)$  denote the values of  $A(x, y)$  and  $B(x, y)$  in front of the lens, in the slab, and behind the lens, respectively. Using the approximations (4.11) and (4.12) we have

$$\left. \begin{aligned} A_m(x, y) &\approx -[\omega(k^e + ik^o)/2]i(\epsilon/2)I(2d - 2d_0 + z), \\ B_m(x, y) &\approx [\omega(k^e - ik^o)/2]i(\epsilon/2)I(2d - 2d_0 + \bar{z}), \\ A_s(x, y) &\approx -[\omega(k^e + ik^o)/2][I(z - 2d_0) + i(\epsilon/2)I(2d - \bar{z})], \\ B_s(x, y) &\approx [\omega(k^e - ik^o)/2][I(\bar{z} - 2d_0) + i(\epsilon/2)I(2d - z)], \\ A_c(x, y) &\approx -[\omega(k^e + ik^o)/2]I(2d - \bar{z}), \\ B_c(x, y) &\approx [\omega(k^e - ik^o)/2]I(2d - z). \end{aligned} \right\} \quad (4.17)$$

From these expressions we see that for fixed  $k^e$  and  $k^o$  the field  $H_Z(x, y)$  is resonant inside two possibly overlapping layers each of thickness  $2(d - d_0)$  one centered at the front interface of the slab and the other centered at the back interface of the slab. Podolskiy *et al.* (2005) had already found that the fields are very large in front of the lens outside the quasistatic regime, and we now see that they become infinitely large as  $\epsilon \rightarrow 0$ .

Substituting the approximation (4.15) into (4.17) yields expressions for  $H_Z(x, y)$  in the resonant regions. In each resonant region

$$H_Z(x, y) \approx G(x, y) \equiv s\omega\{[g^e(z) - g^e(\bar{z})]/2 + [g^o(z) + g^o(\bar{z})]/(2i)\}, \quad (4.18)$$

where  $G(x, y)$  is a piecewise harmonic function of  $x$  and  $y$ , and the prefactor  $s$ , which is 1 inside the lens and  $-1$  outside the lens, is introduced to make the comparison with the quasistatic results easier. One finds that for  $p=e, o$  in the resonant region  $2d_0 - d < x < d_0$  in front of the slab

$$g^p(z) = g_m^p(z) \equiv -ik^p(\epsilon/2)^{(2d_0-d-z)/d} Q_0(2d-2d_0+z), \quad (4.19)$$

where  $q=1$  for  $p=e$  and  $q=-1$  for  $p=o$ , while in the resonant region  $d + d_0 < x < 2d$  behind the slab

$$g^p(z) = g_c^p(z) \equiv k^p(\epsilon/2)^{(z-2d)/d} Q_0(2d-z). \quad (4.20)$$

Within the slab, for  $x < \min\{2d_0, d\}$ , one has the resonant potential

$$g^p(z) = g_{out}^p(z) \equiv -ik^p(\epsilon/2)^{(z-d)/d} Q_0(2d-z), \quad (4.21)$$

which is associated with the front interface and for  $x > \max\{d, 2d_0\}$  one has the resonant potential

$$g^p(z) = g_{in}^p(z) \equiv qk^p(\epsilon/2)^{(2d_0-z)/d} Q_0(z-2d_0), \quad (4.22)$$

which is associated with the back interface, and when  $d_0 < d/2$  for  $2d_0 < x < d$  one has the resonant potential  $g^p(z) = g_{in}^p(z) + g_{out}^p(z)$  where the resonant regions overlap. Here the notations ‘in’ and ‘out’ are introduced to be consistent with the notations in equations (4.8) and (4.9) of MNMP.

The above expressions for  $g_m^p(z)$ ,  $g_c^p(z)$ ,  $g_{out}^p(z)$ , and  $g_{in}^p(z)$  agree precisely with the asymptotic expressions for  $f_{in}^p(z)$ ,  $f_c^p(z)$ ,  $f_{out}^p(z)$ , and  $f_{in}^p(z)$ , respectively, in equations (4.6)–(4.9) of MNMP with the identification  $z = d_0 - z$  corresponding to the different coordinate system used in that paper, with  $\phi=0$  corresponding to the trajectory choice  $\epsilon_s = 1 + i\epsilon$  chosen here, and with the signs of  $k^e$  and  $k^o$  changed due to the  $180^\circ$  rotation associated with the different coordinate system (a dipole rotated by  $180^\circ$  has opposite sign).

Also since  $\epsilon \approx s$  the formula (4.18) is consistent with the formula (2.7) in MNMP for the magnetic field  $H_Z$  once one replaces  $\omega$  with  $-\omega$  because the time dependence in that paper has the factor  $e^{i\omega t}$  rather than  $e^{-i\omega t}$ . It is not surprising that  $H_Z$  becomes asymptotically harmonic in each resonant region as  $\epsilon \rightarrow 0$  since in this limit in the equation  $(\nabla^2 + \omega^2\mu\epsilon)H_Z = 0$  satisfied by  $H_Z$  the spatial derivatives dominate because of the huge gradients in the field  $H_Z$ .

From Maxwell’s equation  $\nabla \times \mathbf{H} = -i\omega\epsilon\mathbf{E}$  we see that in each resonant region

$$E_x(x, y) \approx E_x^r(x, y) \equiv -\partial V(z)/\partial x, \quad E_y(x, y) \approx E_y^r(x, y) \equiv -\partial V(z)/\partial y, \quad (4.23)$$

where

$$V(z) \equiv [g^e(z) + g^e(\bar{z})]/2 + [g^o(z) - g^o(\bar{z})]/(2i). \quad (4.24)$$

In particular, in the resonant region in front of the lens, we have

$$\left. \begin{aligned} E_x^r(x, y) &= -[g_m^e(z) + g_m^e(\bar{z})]/2 - [g_m^o(z) - g_m^o(\bar{z})]/(2i), \\ E_y^r(x, y) &= -i[g_m^e(z) - g_m^e(\bar{z})]/2 - [g_m^o(z) + g_m^o(\bar{z})]/2, \end{aligned} \right\} \quad (4.25)$$

where

$$\begin{aligned} g_m^p(z) &= dg_m^p(z)/dz = i q k^p [(1/d) \log(\epsilon/2)] (\epsilon/2)^{(2d_0-d-z)/d} Q_0(2d-2d_0+z) \\ &\quad - i q k^p (\epsilon/2)^{(2d_0-d-z)/d} Q_0'(2d-2d_0+z) \\ &\approx i q k^p [(1/d) \log(\epsilon/2)] (\epsilon/2)^{(2d_0-d-z)/d} Q_0(2d-2d_0+z), \end{aligned} \quad (4.26)$$

if which we have assumed  $|\log(\epsilon/2)| \gg 1$ . As can be seen from these equations, the fields  $E_x^r(x, y)$  and  $E_y^r(x, y)$  have an approximately exponential decay away from front of the slab face, i.e. they decay as  $e^{x/t}$  with a decay length  $t = d/|\log(\epsilon/2)|$  which depends on  $\epsilon$  and  $d$  but which is *independent* of the frequency  $\omega$  and which is roughly of the order of  $d$  if  $\epsilon$  is not too small. This is in contrast to most evanescent fields which typically have a decay length which is of the order of the wavelength.

The  $Z$ -axis, which is where the dipole source is located, will be in the resonant region when  $d_0 < d/2$  and the resonant field  $(E_x^r, E_y^r) = (E_x^r(0, 0), E_y^r(0, 0)) = (-g_m^e(0), -g_m^o(0))$  acting on it will be given by equation (2.6) with

$$c(\delta) \approx c(\epsilon^2/4) \approx -i[(1/d) \log(\epsilon/2)] (\epsilon/2)^{(2d_0-d)/d} Q_0(2d-2d_0), \quad (4.27)$$

which is in agreement with equation (2.45) when one sets  $\delta = \epsilon^2/4$  and  $\phi = 0$  in accordance with equation (2.2). Now suppose that the source being considered is an electrically polarizable line source satisfying equation (2.7) in which  $(E_x^0, E_y^0)$  is the total field acting on the line source. Also suppose that there are other fixed sources, possibly on both sides of the slab lens, that lie outside the cloaking region, i.e. which are more than a distance  $d/2$  away from the slab. We assume these fixed sources are not perturbed if we remove the polarizable line and we let  $(E_x, E_y)$  denote the field at the  $Z$ -axis due to these sources and the slab lens when the polarizable line source is absent. Then it is easy to check that equations (2.7)–(2.26) remain valid, implying that the polarizable line is cloaked at any frequency, not just in the quasistatic limit, and for very small loss the effective polarizability will be

$$\alpha_* \approx -I/c(\epsilon^2/4) \approx \frac{id(\epsilon/2)^{(d-2d_0)/d}}{|\log(\epsilon/2)| Q_0(2d-2d_0)} I, \quad (4.28)$$

which will be purely imaginary, with a small positive imaginary part, reflecting the loss in the lens due to the localized resonance.

When, for simplicity, the polarizability is proportional to the identity tensor  $\alpha = \alpha I$ , then equations (2.26), (4.19) and (4.27) imply

$$\left. \begin{aligned} g_m^e(z) &\approx \frac{[-E_x + \alpha^{-1} k_0^e] d (\epsilon/2)^{-z/d} Q_0(2d-2d_0+z)}{\log(\epsilon/2) Q_0(2d-2d_0)}, \\ g_m^o(z) &\approx \frac{[-E_y + \alpha^{-1} k_0^o] d (\epsilon/2)^{-z/d} Q_0(2d-2d_0+z)}{\log(\epsilon/2) Q_0(2d-2d_0)}, \end{aligned} \right\} \quad (4.29)$$

which are resonant in the layer between the source and the slab. Similarly by examining  $g_m^e(z)$ ,  $g_{out}^p$  and  $g_{in}^p$  we see that just as in the quasistatic case (see figure 6) the resonance is confined to the two strips  $0 < x < 2d_0$  and  $d < x < d + 2d_0$  each of thickness  $2d_0$  and each with an interface of the slab as its midplane. The energy estimates obtained in §4 of MNMP remain valid: as  $\epsilon \rightarrow 0$  the total electrical energy stored in the slab will scale as

$$[|k^e|^2 + |k^o|^2] \epsilon^{2(d_0/d)-2} |\log \epsilon| \sim \epsilon^{2(d_0/d)-2} |\log \epsilon| / |c(\epsilon^2/4)|^2 \sim \epsilon^{-2d_0/d} / |\log \epsilon|, \quad (4.30)$$

which goes to infinity as  $\epsilon \rightarrow 0$ . Consequently, if the sources are started at some definite time it will take an increasingly long time (but one which is apparently relatively independent of the frequency  $\omega$ ) for the energy in the resonant field to build up to its equilibrium value and for the polarizable line to become cloaked. For fixed but small  $\epsilon$  the transient time will be smallest when  $d_0/d$  is small since then the total electrical energy stored will be dramatically less. Also the cloaking effects will be strongest when  $d_0/d$  is small since then  $c(\epsilon^2/4)$  is largest. Therefore, for cloaking purposes, it is highly advantageous for the polarizable line to be close to the lens. The electrical absorption in the lens will scale like  $\epsilon$  times the above expression, i.e. as  $\epsilon^{(d-2d_0)/d} / |\log \epsilon|$  which goes to zero as  $\epsilon \rightarrow 0$ , and fastest when  $d_0/d$  is small. By a similar analysis, based on equations (4.18), (4.21), and (4.22), the total magnetic field energy  $H_Z$  within the lens scales like

$$\omega^2 [|k^e|^2 + |k^o|^2] \epsilon^{2(d_0/d)-2} / |\log \epsilon| \sim \omega^2 \epsilon^{2(d_0/d)-2} / (|\log \epsilon| |c(\delta)|^2) \sim \omega^2 \epsilon^{-2d_0/d} / |\log \epsilon|^3, \quad (4.31)$$

which for sufficiently small  $\epsilon$  will be much smaller than the electrical energy, but will still go to infinity as  $\epsilon \rightarrow 0$ .

## 5. Cloaking in three dimensions

Since the asymptotic expressions (4.11) for the transfer functions (and the analogous asymptotic expressions for the transfer functions of transverse electric (TE) fields) are independent of the frequency  $\omega$  the three-dimensional cloaking properties and the effective polarizability of a polarizable point dipole in front of the Veselago lens at any fixed frequency should be the same as in the quasistatic limit. Therefore, to simplify the analysis, let us restrict our attention to the quasistatic case, which anyway is more easily experimentally tested since the magnetic permeability can be positive and real everywhere.

We consider the cloaking of a polarizable point dipole in front of a slab of relative permittivity  $\epsilon_s$ . The region in front of the slab has relative permittivity  $\epsilon_m$  and the region behind the slab has relative permittivity  $\epsilon_c$ . We assume that  $\epsilon_c$  and  $\epsilon_m$  remain fixed and that  $\epsilon_s$  approaches  $-\epsilon_m$  along a trajectory in the upper half of the complex plane in such a way that  $\delta \rightarrow 0$  but  $\phi$  remains fixed, where  $\delta$  and  $\phi$  are given by equation (2.4). It proves convenient to use  $x_1$ ,  $x_2$  and  $x_3$  as our coordinates, rather than  $x$ ,  $y$  and  $Z$ , with the polarizable dipole being at  $\mathbf{x} = 0$  and the slab faces being located at  $x_1 = d_0$  and at  $x_1 = d_0 + d$ .

By differentiating with respect to  $x_1$ ,  $x_2$ , and  $x_3$  the plane wave expansion for the potential associated with a suitably normalized point charge,

$$\frac{1}{r} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_2 dk_3 \frac{e^{-\kappa x_1 + i(k_2 x_2 + k_3 x_3)}}{\kappa} \quad \text{for } x_1 > 0, \text{ where } \kappa = \sqrt{k_2^2 x_2 + k_3^2 x_3}, \quad (5.1)$$

one obtains the plane wave expansion for a dipole

$$-\mathbf{a} \cdot \nabla [1/(4\pi r)] = (\mathbf{a} \cdot \mathbf{x})/(4\pi r^3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_2 dk_3 a(k_2, k_3) e^{-\kappa x_1 + i(k_2 x_2 + k_3 x_3)}, \quad (5.2)$$

in which

$$\mathbf{a} = (a_1, a_2, a_3), \quad a(k_2, k_3) = [a_1 - i a_2(k_2/\kappa) - i a_3(k_3/\kappa)]/(8\pi^2), \quad (5.3)$$

and  $a_1$ ,  $a_2$  and  $a_3$  are (apart from a constant factor) the possibly complex strengths of the dipole components in the  $x_1$ ,  $x_2$ , and  $x_3$  directions.

By the superposition principle the potential  $V(\mathbf{x})$  in the slab geometry is given by the expression

$$V(\mathbf{x}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dk_2 dk_3 a(k_2, k_3) \tau(\mathbf{x}; k_2, k_3), \quad (5.4)$$

where the transfer function  $\tau(\mathbf{x}; k_2, k_3)$  represents the solution for  $V$  with an incident field  $V^{\text{inc}} = e^{-\kappa x_1 + i(k_2 x_2 + k_3 x_3)}$ . Let  $\tau_m(\mathbf{x}; k_2, k_3)$ ,  $\tau_s(\mathbf{x}; k_2, k_3)$  and  $\tau_c(\mathbf{x}; k_2, k_3)$  denote the expressions for  $\tau(\mathbf{x}; k_2, k_3)$  in front (to the left) of the slab lens, in the slab lens, and behind (to the right) of the slab lens, respectively. These have the form

$$\left. \begin{aligned} \tau_m(\mathbf{x}; k_2, k_3) &= e^{-\kappa x_1 + i(k_2 x_2 + k_3 x_3)} + \beta_m(\kappa) e^{\kappa x_1 + i(k_2 x_2 + k_3 x_3)}, \\ \tau_s(\mathbf{x}; k_2, k_3) &= \alpha_s(\kappa) e^{-\kappa x_1 + i(k_2 x_2 + k_3 x_3)} + \beta_s(\kappa) e^{\kappa x_1 + i(k_2 x_2 + k_3 x_3)}, \\ \tau_c(\mathbf{x}; k_2, k_3) &= \alpha_c(\kappa) e^{-\kappa x_1 + i(k_2 x_2 + k_3 x_3)}. \end{aligned} \right\} \quad (5.5)$$

The requirements of continuity of the potential  $\tau$  and normal component of the associated displacement field  $-\varepsilon \partial \tau / \partial x$  at the interfaces  $x = d_0$  and  $x = d + d_0$  determine the coefficients

$$\left. \begin{aligned} \beta_m(\kappa) &= \frac{(e^{2(d-d_0)\kappa} / \eta_{\text{sc}}) - \delta e^{i\phi} \eta_{\text{sc}} e^{-2d_0\kappa}}{1 + \delta e^{i\phi} e^{2d\kappa}}, & \alpha_s(\kappa) &= \frac{2\varepsilon_m e^{2d\kappa}}{\eta_{\text{sc}}(\varepsilon_m - \varepsilon_s)(1 + \delta e^{i\phi} e^{2d\kappa})}, \\ \beta_s(\kappa) &= \frac{2\varepsilon_m e^{-2d_0\kappa}}{(\varepsilon_m - \varepsilon_s)(1 + \delta e^{i\phi} e^{2d\kappa})}, & \alpha_c(\kappa) &= \frac{4\varepsilon_s \varepsilon_m e^{2d\kappa}}{(\varepsilon_s - \varepsilon_c)(\varepsilon_m - \varepsilon_s)(1 + \delta e^{i\phi} e^{2d\kappa})}, \end{aligned} \right\} \quad (5.6)$$

where  $\eta_{\text{sc}} = (\varepsilon_s - \varepsilon_c)/(\varepsilon_s + \varepsilon_c)$ . Introducing the angle  $\varphi$  such that  $k_2 = \kappa \cos \varphi$ ,  $k_3 = \kappa \sin \varphi$ , then the integral (5.4) becomes

$$V(\mathbf{x}) = \int_0^{2\pi} d\varphi a(\varphi) \int_0^{\infty} d\kappa \kappa \alpha(\kappa) e^{\kappa(-x_1 + i x_2 \cos \varphi + i x_3 \sin \varphi)} + \kappa \beta(\kappa) e^{\kappa(x_1 + i x_2 \cos \varphi + i x_3 \sin \varphi)}, \quad (5.7)$$

in which  $\alpha(\kappa)$  equals 1,  $\alpha_s(\kappa)$  and  $\alpha_c(\kappa)$ , in front of the lens, in the lens, and behind the lens, respectively, and similarly  $\beta(\kappa)$  equals  $\beta_m(\kappa)$ ,  $\beta_s(\kappa)$  and 0, in these respective regions, and where

$$a(\varphi) = (a_1 - ia_2 \cos \varphi - ia_3 \sin \varphi) / (8\pi^2). \quad (5.8)$$

Let us define

$$J(b) \equiv \int_0^\infty d\kappa \frac{\kappa e^{\kappa b}}{1 + \delta e^{i\phi} e^{2d\kappa}}, \quad J'(b) \equiv \frac{dJ(b)}{db} = \int_0^\infty d\kappa \frac{\kappa b e^{\kappa b}}{1 + \delta e^{i\phi} e^{2d\kappa}}. \quad (5.9)$$

Then we have

$$\left. \begin{aligned} V_m(\mathbf{x}) &= (\mathbf{a} \cdot \mathbf{x}) / 4\pi r^3 + \int_0^{2\pi} d\varphi a(\varphi) [J(2d - 2d_0 + x_1 + ix_2 \cos \varphi + ix_3 \sin \varphi) / \eta_{sc} \\ &\quad - \delta e^{i\phi} \eta_{sc} J(-2d_0 + x_1 + ix_2 \cos \varphi + ix_3 \sin \varphi)], \\ V_s(\mathbf{x}) &= \int_0^{2\pi} d\varphi a(\varphi) [J(2d - x_1 + ix_2 \cos \varphi + ix_3 \sin \varphi) / \eta_{sc} \\ &\quad + J(-2d_0 + x_1 + ix_2 \cos \varphi + ix_3 \sin \varphi)] [2\varepsilon_m / (\varepsilon_m - \varepsilon_s)], \\ V_c(\mathbf{x}) &= \int_0^{2\pi} d\varphi a(\varphi) J(2d - x_1 + ix_2 \cos \varphi + ix_3 \sin \varphi) (4\varepsilon_s \varepsilon_m) / [(\varepsilon_s - \varepsilon_c)(\varepsilon_m - \varepsilon_s)]. \end{aligned} \right\} \quad (5.10)$$

The electric field in front of the lens will be

$$\mathbf{E}(\mathbf{x}) = -\nabla V_m = \mathbf{E}^{\text{dip}}(\mathbf{x}) + \mathbf{E}^r(\mathbf{x}), \quad (5.11)$$

where  $\mathbf{E}^{\text{dip}}$  is the field due to the dipole alone,

$$\mathbf{E}^{\text{dip}} = -\nabla [\mathbf{a} \cdot \mathbf{x} / (4\pi r^3)] = [3\mathbf{x}(\mathbf{a} \cdot \mathbf{x}) / r^5 - \mathbf{a} / (4\pi r^3)], \quad (5.12)$$

and  $\mathbf{E}^r(\mathbf{x})$  is the response field (which in the limit as  $\delta \rightarrow 0$  can become the resonant field) given by

$$\begin{aligned} \mathbf{E}^r(\mathbf{x}) &= (E_1^r, E_2^r, E_3^r) \\ &= \int_0^{2\pi} d\varphi a(\varphi) (1, i \cos \varphi, i \sin \varphi) [-J'(2d - 2d_0 + x_1 + ix_2 \cos \varphi + ix_3 \sin \varphi) / \eta_{sc} \\ &\quad + \delta e^{i\phi} \eta_{sc} J'(-2d_0 + x_1 + ix_2 \cos \varphi + ix_3 \sin \varphi)]. \end{aligned} \quad (5.13)$$

In particular at the origin  $\mathbf{x} = 0$ , which is where the dipole source is located, the integral is easily calculated and we have

$$\mathbf{E}^r(0) = c(\delta) \mathbf{L} \mathbf{a} \quad \text{with} \quad \mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}, \quad (5.14)$$

where

$$c(\delta) = -J'(2d - 2d_0) / (4\pi \eta_{sc}) + \delta e^{i\phi} \eta_{sc} J'(-2d_0) / (4\pi). \quad (5.15)$$

So far no approximation has been made.

Let us now examine  $J(b)$  for complex values of  $b = b' + ib''$  in the limit as  $\delta \rightarrow 0$ . If  $b'$  is negative then we have the estimate

$$|J(b)| \leq \int_0^\infty d\kappa \kappa |e^{\kappa b}| / c_0 = 1/(c_0 b'^2), \quad \text{where } c_0 = \min_{c \geq 0} |1 + ce^{i\phi}|, \quad (5.16)$$

and  $c_0$  is non-zero because  $\phi$  is never equal to  $\pi$  or  $-\pi$ . So in this case  $J(b)$  remains bounded as  $\delta \rightarrow 0$ , and

$$\lim_{\delta \rightarrow 0} J(b) = \int_0^\infty d\kappa \lim_{\delta \rightarrow 0} \frac{\kappa e^{\kappa b}}{1 + \delta e^{i\phi} e^{2d\kappa}} = \int_0^\infty d\kappa \kappa e^{\kappa b} = 1/b^2. \quad (5.17)$$

It follows from equation (5.10) that when  $\varepsilon_c \neq \varepsilon_m$  and  $\mathbf{a}$  is fixed the potential  $V(\mathbf{x})$  is not resonant outside the layer  $2d > x_1 > 2(d_0 - d)$ .

When  $b' > 0$  let the transition point  $k_t$  be defined by  $\delta e^{2dk_t} = 1$ , i.e. by  $k_t = -(1/2d) \log \delta$ , and let us change the variable of integration from  $\kappa$  to  $v = \kappa - k_t$ . Then the integral becomes

$$J(b) = \int_{-k_t}^\infty dv \frac{(v + k_t)e^{(v+k_t)b}}{1 + e^{2dv}} \approx \int_{-\infty}^\infty dv \frac{(v + k_t)e^{(v+k_t)b}}{1 + e^{2dv}}, \quad (5.18)$$

and this latter integral can be expressed in terms of the functions  $Q(b)$ , given by equation (2.42), and its derivative  $Q'(b) = dQ(b)/db$ :

$$\begin{aligned} \int_{-\infty}^\infty dv \frac{(v + k_t)e^{(v+k_t)b}}{1 + e^{2dv}} &= \frac{d}{db} \int_{-\infty}^\infty dv \frac{e^{(v+k_t)b}}{1 + e^{2dv}} = \frac{d}{db} [e^{k_t b} Q(b)] \\ &= k_t e^{k_t b} Q(b) + e^{k_t b} Q'(b) \\ &\approx -[(1/2d) \log \delta] \delta^{-b/2d} Q(b), \end{aligned} \quad (5.19)$$

where we have assumed  $Q(b) \neq 0$  and used the fact that  $k_t$  is large for extremely small  $\delta$ . Combining formulae gives

$$\left. \begin{aligned} J(b) &\approx -[(1/2d) \log \delta] \delta^{-b/2d} Q(b), \\ J'(b) &= dJ(b)/db \approx [(1/2d) \log \delta]^2 \delta^{-b/2d} Q(b) - [(1/2d) \log \delta] \delta^{-b/2d} Q'(b) \\ &\approx (\log \delta)^2 \delta^{-b/2d} Q(b)/(4d^2), \end{aligned} \right\} \quad (5.20)$$

and so for  $d_0 < d$  when  $\delta$  is very small

$$c(\delta) \approx -J'(2d - 2d_0)/(4\pi\eta_{sc}) \approx -(\log \delta)^2 \delta^{(d_0-d)/d} Q(2d - 2d_0)/(16\pi d^2 \eta_{sc}). \quad (5.21)$$

If  $\varepsilon_c \neq \varepsilon_m$  and  $\delta \rightarrow 0$  then  $\eta_{sc} \rightarrow 1/\eta$  and the above expression gives

$$c(\delta) \approx -\eta (\log \delta)^2 \delta^{(d_0-d)/d} Q(2d - 2d_0)/(16\pi d^2), \quad (5.22)$$

which implies the dipole becomes cloaked ( $|c(\delta)|$  is large) for all  $d_0 < d$ . In the case when  $\varepsilon_c = \varepsilon_m$  then  $\eta_{sc} \approx i e^{-i\phi/2} / \sqrt{\delta}$  for small  $\delta$  and we have

$$c(\delta) \approx i e^{i\phi/2} (\log \delta)^2 \delta^{(2d_0-d)/2d} Q(2d-2d_0)/(16\pi d^2), \quad (5.23)$$

which implies the dipole becomes cloaked for all  $d_0 < d/2$ .

To justify these cloaking claims suppose the dipole at  $\mathbf{x} = 0$  is a polarizable ‘molecule’ in the cloaking region. Let  $\mathbf{E}(\mathbf{x})$  be the field without the polarizable dipole present due to the following.

- (i) Fields generated by fixed quasistatic sources lying outside the slab (on either side of it) and which are outside the cloaking layer.
- (ii) Fields generated by the slab due to its interaction with these fixed sources.

The field  $\mathbf{E}^0$  acting on the polarizable molecule has two components:

$$\mathbf{E}^0 = \mathbf{E}(0) + \mathbf{E}^r(0) = \mathbf{E}(0) + c(\delta) \mathbf{L} \mathbf{a}, \quad (5.24)$$

where  $\mathbf{E}^r(0)$  is the field due to the interaction of the polarizable molecule with the slab. If  $\boldsymbol{\alpha}$  denotes the polarizability tensor of the molecule (which need not be proportional to  $\mathbf{I}$ ) and we allow for the fact that the polarizable line could have a fixed dipole source term  $\mathbf{a}_0$  then we have

$$\mathbf{a} = \boldsymbol{\alpha} \mathbf{E}^0 + \mathbf{a}_0 = \boldsymbol{\alpha}_* \mathbf{E}(0) + \mathbf{a}_*, \quad (5.25)$$

where

$$\boldsymbol{\alpha}_* = [\boldsymbol{\alpha}^{-1} - c(\delta) \mathbf{L}]^{-1}, \quad \mathbf{a}_* = [\mathbf{I} - c(\delta) \boldsymbol{\alpha} \mathbf{L}]^{-1} \mathbf{a}_0, \quad (5.26)$$

are the ‘effective polarizability tensor’ and ‘effective source term’. Notice that the ‘effective polarizability tensor’ is anisotropic when  $\boldsymbol{\alpha} = \alpha \mathbf{I}$ .

It is interesting to examine what happens if the polarizable ‘molecule’ is behind the slab. When  $\varepsilon_c = \varepsilon_m$  symmetry considerations imply that the molecule will be cloaked if it is within a distance  $d/2$  from the slab. When  $\varepsilon_c \neq \varepsilon_m$  we will see that a molecule behind the slab will never be cloaked. To establish the latter it suffices to consider the situation where the molecule is in front of the lens and  $\varepsilon_s$  tends to  $-\varepsilon_c$ . Then equation (2.1) implies  $\varepsilon_s \approx (-1 - 2\delta e^{i\phi}/\eta) \varepsilon_c$  and  $\eta_{sc} \approx \eta e^{-i\phi}/\delta$ . It follows from equation (5.15) that

$$c(\delta) \approx [-\delta e^{i\phi} J'(2d-2d_0)/\eta + \eta J'(-2d_0)]/(4\pi). \quad (5.27)$$

Now as  $\delta \rightarrow 0$  equation (5.16) implies  $J'(-2d_0)$  remains bounded, while equation (5.20) implies  $\delta J'(2d-2d_0)$  scales as  $(\log \delta)^2 \delta^{d_0/d}$ . Therefore  $c(\delta)$  remains bounded and no cloaking occurs.

## 6. Not everything is cloaked

Here we show that a layer of permittivity  $\varepsilon_2 \neq \varepsilon_m$  is not cloaked when it is inserted in the cloaking region. Let us consider the quasistatic transfer function  $\tau(\mathbf{x}; k_2, k_3)$  associated with a multilayered system with interfaces at  $x_1 = s_1, s_2, s_3, s_4$ , where  $0 < s_1 < s_2 < s_3 < s_4$  and in the region between  $s_j$  and  $s_{j+1}$  the permittivity is  $\varepsilon_{j+1}$ , while for  $x_1 < s_1$  it is  $\varepsilon_1$  and for  $x > s_4$  it is  $\varepsilon_5$ . In the region

occupied by the material with permittivity  $\varepsilon_j$ , for  $j=1,2,\dots,5$  the transfer function takes the form

$$\tau_j(\mathbf{x}; k_2, k_3) = \alpha_j e^{-\kappa x_1 + i(k_2 x_2 + k_3 x_3)} + \beta_j e^{\kappa x_1 + i(k_2 x_2 + k_3 x_3)}, \quad (6.1)$$

where  $\alpha_1 = 1$  and  $\beta_5 = 0$ . The requirements of continuity of the potential  $\tau$  and normal component of the associated displacement field  $-\varepsilon \partial \tau / \partial x$  at the interface  $x_1 = s_j$  imply

$$\begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix} = \mathbf{M}_j \begin{pmatrix} \alpha_{j+1} \\ \beta_{j+1} \end{pmatrix}, \quad \mathbf{M}_j \equiv \frac{1}{2} \begin{pmatrix} 1 + \varrho_j & (1 - \varrho_j) e^{2s_j \kappa} \\ (1 - \varrho_j) e^{-2s_j \kappa} & 1 + \varrho_j \end{pmatrix}, \quad \varrho_j \equiv \frac{\varepsilon_{j+1}}{\varepsilon_j}. \quad (6.2)$$

Thus we have

$$\begin{pmatrix} 1 \\ \beta_1 \end{pmatrix} = \mathbf{M} \begin{pmatrix} \alpha_5 \\ 0 \end{pmatrix}, \quad \text{where } \mathbf{M} = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} = \mathbf{M}_1 \mathbf{M}_2 \mathbf{M}_3 \mathbf{M}_4, \quad (6.3)$$

which implies  $\alpha_5 = 1/m_1$  and  $\beta_1 = m_3/m_1$ . Now let us take

$$\varepsilon_1 = \varepsilon_3 = \varepsilon_5 = \varepsilon_m = 1, \quad \varepsilon_4 = \varepsilon_s = -1 + i\epsilon, \quad s_3 = d_0, \quad s_4 = d + d_0, \quad (6.4)$$

where  $\epsilon$  is very small, so that the multilayered system consists of a layer of permittivity  $\varepsilon_2$  and thickness  $s_2 - s_1$  in front of a superlens and not touching it. Then we have

$$\lim_{\epsilon \rightarrow 0} \mathbf{M} = \mathbf{M}_1 \mathbf{M}_2 \begin{pmatrix} e^{-2d\kappa} & 0 \\ 0 & e^{2d\kappa} \end{pmatrix},$$

and consequently  $\alpha_5$  and  $\beta_1$  still depend on  $\varepsilon_2$  even if the layer of permittivity  $\varepsilon_2$  lies entirely inside the cloaking region: thus the presence of the layer can be detected from outside the cloaking region.

On the basis of this example one might think that cloaking does not extend to bodies of arbitrary shape. However, the example is very special in that the potential in the region  $s_2 < x_1 < s_0$  when analytically extended into the region  $x_1 \leq s_2$  has no singularities there, and in particular no singularities in the cloaking region. By contrast, any object of finite extent lying entirely within the cloaking region of the slab lens will have singularities in the analytic continuation of the potential outside the object to the region within the object. In order for these singularities not to create resonant regions with infinite energy in the limit  $\epsilon \rightarrow 0$  it seems plausible that their effects should diminish as  $\epsilon \rightarrow 0$ . Therefore it may be the case that any object of finite extent lying entirely within the cloaking region of the slab lens will be cloaked in the limit  $\epsilon \rightarrow 0$ . It seems much more speculative to suggest that an object lying half way in the cloaking region would be half cloaked. However, if this were true it could provide an interesting way to image the interior of an object: when one looked through the slab lens at the object one would only see only the back half of it!

To experimentally detect cloaking it may be necessary to use lasers to provide coherent radiation. If ordinary electromagnetic radiation were used then by the time the resonant field responsible for the cloaking reached an equilibrium value

the radiation acting on the polarizable ‘molecule’ could be out of coherence with the resonant field acting on it. This might not be the case if the ‘molecule’ is sufficiently close to the boundary since then the resonant field reaches its equilibrium value relatively quickly. Also we have assumed that the ‘molecule’ is a stationary object, and therefore it seems possible that thermal effects could destroy cloaking. To minimize these effects it may be necessary to reduce the temperature as much as possible. Moreover, it seems unlikely that cloaking could be achieved over a broad range of frequencies since if  $\epsilon_s \approx -1$  at one frequency and we assume the loss terms are extremely small, the positivity of  $d(\omega\epsilon_s)/d\omega$  (which enters the Brillouin formula for the energy: see §80 of Landau & Lifshitz (1960)) implies that  $d\epsilon_s/d\omega > 1/\omega$ , so there is necessarily a significant change of the permittivity  $\epsilon_s$  with frequency.

It should also be cautioned that we have not investigated the effects of stability. If the time harmonic solution assumed here is unstable it seems unlikely that cloaking could be experimentally observed.

The authors thank Alexei Efros for his insightful remarks and for his suggestion, noted in the paper MNMP, that the energy absorption in the slab lens may be infinite for a fixed source close to the lens. Also they wish to acknowledge that Arthur Yaghjian independently discovered that the field diverges in front of the Veselago lens (2005, personal communication). Viktor Podolskiy is thanked for discussions regarding the resonant region in front of the slab lens and for providing the asymptotic formula for the transfer function in front of the lens. Ross McPhedran is thanked for helpful conversations. The referees are thanked for helpful comments. G.W.M. is grateful for support from the National Science Foundation through grant DMS-0411035, and for the hospitality of Sydney University during his stay there. N.A.P.N. is grateful for support from the University of Technology, Sydney.

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